Recall: a real-life scenario

A company with 100 employees has six projects running simultaneously, each having its own leader. Each project leader wants to schedule a one hour project meeting, but since an employee might be part of several projects and each project member should be present at each relevant meeting, the scheduling is problematic.

The company rents an office building to accommodate the meetings and wants to *minimize the cost*. There are several rooms in the building, so meetings can take place parallel, but rooms cannot be rented separately.

The company requests project leaders to be available between 8-10 and tries to schedule a conflict-free project meeting schedule by finding a *proper coloring* of the conflict graph of the projects, using the timeslots 8-9 and 9-10 as colors.

Recall: Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $c : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring.

Complexity: 2-*colorability* is in P, but 3-*colorability* is NP-complete (reduce to 3-SAT; HW)
k-colorability is in NP, but it is not known to be in co-NP for $k \geq 3$.
Want: sufficient conditions guaranteeing non-k-colorability
$\rightsquigarrow$ search for lower bounds on $\chi(G)$

Recall: Lower bounds and their tightness

Observation For every graph G, $\chi(G) \geq \omega(G)$.

- when $\chi(G) = \omega(G)$:

  - cliques, bipartite graphs, complement of bipartite graphs, interval graphs, perfect graphs

- when $\chi(G) > \omega(G)$:

  - odd cycles, complement of odd cycles (Perfect Graph Theorem!)
  - *Mycielski's Construction*: graphs with $\omega(G) = 2$ and $\chi(G) > 10^{10^{10}}$
  - *Typical behaviour:* For the uniform random graph $G = G(n, \frac{1}{2})$,

$$\begin{aligned} \chi(G) &\approx \frac{n}{2\log_2 n} \\ \omega(G), \alpha(G) &\approx 2\log_2 n \end{aligned}$$

A typically asymptotically tight lower bound.
For every graph G, we have

$$\chi(G) \geq \frac{v(G)}{\alpha(G)}.$$

Forced subdivision

G contains a $K_k \Rightarrow \chi(G) \geq k$

G contains a $K_k \not\Leftarrow \chi(G) \geq k$ (already for $k \geq 3$)

... and the uphill battle for useful/nice necessary conditions for an NP-complete property starts ...

Hajós' Conjecture (1961)

G contains a K_k-subdivision $\overset{?}{\Leftarrow} \chi(G) \geq k$

An H-subdivision is a graph obtained from H by successive edge-subdivisions.

Remark. The conjecture is true for $k = 2$ and $k = 3$.

Theorem (Dirac, 1952) Hajós' Conjecture is true for $k = 4$.

Counterexample (Catlin, 1979)
Hajós' Conjecture is false for $k \geq 7$. (HW)

Hadwiger's Conjecture (1943)

G contains a K_k-*minor* $\overset{?}{\Leftarrow} \chi(G) \geq k$

Proved for $k \leq 6$. Open for $k \geq 7$.

Proof of Dirac's Theorem

Theorem (Dirac, 1952) If $\chi(G) \geq 4$ then G contains a K_4-subdivision.

Proof. Induction on $v(G)$. $v(G) = 4 \Rightarrow G = K_4$.

W.l.o.g. G is 4-critical.

Case 0. $\kappa(G) = 0$ would contradict 4-criticality

Case 1. $\kappa(G) = 1$ would contradict 4-criticality

Case 2. $\kappa(G) = 2$. Let $S = \{x, y\}$ be a cut-set.

$xy \in E(G)$ would contradict 4-criticality

Hence $xy \notin E(G)$.

$\chi(G) \geq 4 \Rightarrow G$ must have an S-lobe H, such that $\chi(H+xy) \geq 4$. Apply induction hypothesis to $H+xy$ and find a K_4-subdivision F in $H + xy$. Then modify F to obtain a K_4-subdivision in G.

Let $S \subseteq V(G)$. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$.

Proof of Dirac's Theorem— Continued

Case 3. $\kappa(G) \geq 3$. Let $x \in V(G)$. $G - x$ is 2-connected, so contains a cycle C.

Claim. There is an x, C-fan of size 3.

Proof. Add a new vertex u to G connecting it to the vertices of C. By the Expansion Lemma the new graph G' is 3-connected. By Menger's Theorem there exist three p.i.d x, u-paths P_1, P_2, P_3 in G'. $\square$

Given a vertex x and a set U of vertices, and x, U-fan is a set of paths from x to U such that any two of them share only the vertex x.
Fan Lemma. G is k-connected iff $|V(G)| \geq k + 1$ and for every choice of $x \in V(G)$ and $U \subseteq V(G)$, $|U| \geq k$, G has an x, U-fan.

Then $C \cup P_1 \cup P_2 \cup P_3 - u$ is K_4-subdivision in G.

Upper bounds: tightness

Recall: Proposition $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic; Greedy coloring (Order vertices arbitrarily; color in this order with first available color) $\square$

Recall: Definition A graph is called r-degenerate, if there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that for every $i = 1, 2, \ldots, n-1$, we have

$$|N(v_{i+1}) \cap |\{v_1, \ldots, v_i\}| \leq r.$$

Recall: Theorem. G is r-degenerate $\Rightarrow$ $\chi(G) \leq r + 1$.

Proof. Greedy coloring using the special vertex order.

Brooks' Theorem. (1941) Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ iff G is a complete graph or an odd cycle.

Proof. Trickier, but still greedy coloring...
order: follow spanning tree from leaves to root

Proof of Brooks' Theorem. Cases.

Case 1. G is not regular.
Let the root be a vertex with degree $< \Delta(G)$.

Case 2. G has a cut-vertex.
Let the root be the cut-vertex.

Assume G is k-regular and $\kappa(G) \geq 2$.

Case 3. $k \leq 2$. Then $G = C_l$ or K_2.

Assume $k \geq 3$. We need a root v_n with nonadjacent neighbors v_1, v_2, such that $G - \{v_1, v_2\}$ is connected. Let x be a vertex of degree less than $v(G) - 1$.

Case 4. $\kappa(G - x) \geq 2$.
Let v_n be a neighbor of x, which has a neighbor y, such that y and x are non-neighbors. Then let $v_1 = x$ and $v_2 = y$.

Case 5. $\kappa(G - x) = 1$.
Then x has a neighbor in every leaf-block of $G - x$. Let $v_n = x$ and v_1, v_2 be two neighbors of x in different leaf blocks of $G - x$.

Block-decomposition of connected graphs

Maximal induced subgraph of G with no cut-vertex is called block of G.

Lemma. Two blocks intersect in at most one vertex.

Proof. If B_1 and B_2 have no cut-vertex and share at least two vertices then $B_1 \cup B_2$ has no cut-vertex either.

The Block/Cut-vertex graph of G is a bipartite graph with vertex set

$$\{B : B \text{ is a block}\} \cup \{v : v \text{ is a cut-vertex}\}.$$

Block B is adjacent to cut-vertex v iff $v \in V(B)$.

Proposition. The Block/Cut-vertex graph of a connected graph is a tree.

A more complicated real-life scenario

Project leaders are very important and very busy and not so flexible to be available at wish of the company; they want to identify the possible one-hour-slots themselves. One might want to be available 8-10, the other 9-11, the third one 8-9 and 10-11, etc.

Is a conflict-free scheduling still possible? Or the administration should ask project leaders to be available for more than just two one-hour timeslots? How many should they ask for?

This scenario, when each vertex (project) has its own set of available colors (timeslots) is the setting of *list coloring*.

List Coloring

$v \in V(G)$, $L(v)$ a list of colors
A list coloring is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$.

G is k-choosable or k-list-colorable if **every** assignment of k-element lists permits a proper coloring.

$\chi_l(G) = \min\{k : G \text{ is } k\text{-choosable}\}$

Claim $\chi_l(G) \geq \chi(G)$

Example: $K_{2,2}$
Example: $\chi_l(K_{3,3}) \neq \chi(K_{3,3})$

Claim $\chi_l(G) \leq \Delta(G) + 1$

Example: $\chi_l(G) - \chi(G)$ can be arbitrary large:
Proposition $K_{m,m}$ is not k-choosable for $m = \binom{2k-1}{k}$.

Complexity: It is unclear whether k-*choosability* is in NP and it is also unclear whether it is in co-NP. So it is “more difficult” than k-*colorability*!

Recall: Four-Color Theorem (Appel-Haken, 1976)
Every planar graph is 4-colorable.
Proof: Very-very long, tedious.

Recall: Five-Color Theorem (Heawood, 1890)
Every planar graph is 5-colorable.
Proof: Proved in Discrete Math I.

HW. There is a planar graph which is not 4-list-colorable. (Voigt, Mirzakhani)

Theorem. (Thomassen) Every planar graph G is 5-list colorable.

Stronger Statement. Let G be a plane graph with an outer face bounded by cycle C. Suppose that
- two vertices v_1, v_2, $v_1v_2 \in E(C)$ are colored by two different colors,
- the other vertices of C have 3-element lists assigned to them and
- the internal vertices have 5-element lists assigned to them.

Then the coloring of v_1 and v_2 can be extended properly to the whole G using colors from the assigned lists for each vertex.

Proof.. W.l.o.g. every face of G is a triangle, except maybe the outer face.

Induction on $v(G)$. For $v(G) = 3$, $G = K_3$, OK.

For $v(G) > 3$, there are two cases.

Case 1. There is a chord $v_i v_j$ of C.
Cut to two smaller graphs along the chord, color first the piece where both v_1 and v_2 lie, then color the other piece.

Case 2. C has no chord.
Designate two colors $x, y \in L(v_3)$ such that they differ from the color of v_2. Color $G - v_3$ by induction (boundary is a cycle!), such that x and y are deleted from the lists of the interior neighbors of v_3. Extend the coloring to v_3.

Edge coloring

A k-edge-coloring of a multigraph G is a function $c : E(G) \to S$, where $|S| = k$. The k-edge-coloring is proper if incident edges have different c-values (colors). A multigraph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number (or chromatic index) of a loopless multigraph G is

$$\chi'(G) := \min\{k : \ G \text{ is } k\text{-edge-colorable}\}.$$

Examples. K_4, K_5, K_n, $\Delta(G) \leq \chi'(G)$

Motivation. Efficient round-robin tournament scheduling.

Observation The color classes of a proper edge coloring are matchings. In particular, if G is regular and $\Delta(G) = \chi'(G)$, then G has a perfect matching.

Theorem. (König, 1916)
For a bipartite multigraph G, $\chi'(G) = \Delta(G)$

Proposition. $\chi'(Petersen) = 4$.

Line graphs and Vizing's Theorem

Line graph $L(G)$: vertex set $V(L(G)) = E(G)$ and edge set $E(L(G)) = \{ef : e \cap f \neq \emptyset\}$

Observations

- $M \subseteq E(G)$ is a matching $\Leftrightarrow$ $M \subseteq V(L(G))$ is an independent set
- $c : E(G) \to [k]$ is a proper edge-coloring of G $\Leftrightarrow$ $c : V(L(G)) \to [k]$ is a proper vertex-coloring of $L(G)$
- Hence $\chi'(G) = \chi(L(G))$, so

$$\begin{array}{rcccl} \Delta(G) & \leq & \omega(L(G)) & & \\ & \leq & \chi'(G) & \leq & \Delta(L(G)) + 1 \\ & & & \leq & 2\Delta(G) - 1 \end{array}$$

Theorem. (Vizing, 1964) For a simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Generalization. If the maximum edge-multiplicity in a multigraph G is $\mu(G)$, then $\chi'(G) \leq \Delta(G) + \mu(G)$

Example. Fat triangle; $\chi'(G) = \Delta(G) + \mu(G)$.

Proof of Vizing's Theorem (A. Schrijver)

Induction on $v(G)$.

If $v(G) = 1$, then $G = K_1$; the theorem is OK.

Assume $v(G) > 1$. Delete a vertex $v \in V(G)$. By induction $G - v$ is $(\Delta(G) + 1)$-edge-colorable.

Why is G also $(\Delta(G) + 1)$-edge-colorable?

We prove the following

Stronger Statement. Let G be a simple graph and $k \geq 1$ be an integer. Let $v \in V(G)$, such that

- $d(v) \leq k$,
- $d(u) \leq k$ for every $u \in N(v)$, and
- $d(u) = k$ for at most one $u \in N(v)$.

Then
$G - v$ is k-edge-colorable $\Rightarrow$ G is k-edge-colorable.

Proof of the Stronger Statement I

Induction on k (!!!)

For $k = 1$ it is OK.

W.l.o.g. $d(u) = k - 1$ for every $u \in N(v)$, except for *exactly one* $w \in N(v)$ for which $d(w) = k$.

Let $c : E(G - v) \to \{1, \ldots, k\}$ be a proper k-edge-coloring of $G - v$, which minimizes*

$$\sum_{i=1}^{k} |X_i|^2,$$

where $X_i := \{u \in N(v) : \ u \text{ is missing color } i\}$.

*I.e., we choose the coloring so the $|X_i|$s "as equal as possible".

Proof of the Stronger Statement II

Case 1. There is an i, with $|X_i| = 1$. Say $X_k = \{u\}$.

Let $G' = G - uv - \{xy : c(xy) = k\}$.
Apply the induction hypothesis for G', $k-1$, and v.

Case 2. $|X_i| \neq 1$ for every $i = 1, \ldots, k$.

Then

$$\sum_{l=1}^{k} |X_l| = 2d(v) - 1 < 2k.$$

So there are colors i with $|X_i| = 0$ and
j with $|X_j| \geq 3$.

Let $H \subseteq G$ be subgraph spanned by the edges of color i and j.
Switch colors i and j in a component C of H, where $C \cap X_i \neq \emptyset$.
This reduces $\sum_{l=1}^{k} |X_l|^2$, a contradiction. $\square$

Edge-List Coloring

List Coloring Conjecture (1985) $\chi'_l(G) = \chi'(G)$

HW (from last week) True when G is a cycle.

Greedy Coloring $\chi'_l(G) \leq 2\chi'(G) - 1$.

Theorem (Kahn, 1996) $\chi'_l(G) = \chi'(G)(1 + o(1))$

Proof: probabilistic, difficult

Theorem (Galvin, 1995) For any bipartite graph B,

$$\chi'_l(B) = \chi'(B).$$

We prove Galvin's Theorem only for $B = K_{n,n}$ (which was known as the **Dinitz Conjecture** since 1979)

HW: Modify proof for arbitrary bipartite graph B.

Recall $\chi'(K_{n,n}) = n$. So, no matter how each edge of $K_{n,n}$ gets assigned a list of n colors, we should find a proper coloring of the edges from their lists.

First we distill important structural information about greedy colorings, so to accommodate a tricky inductive argument.

Kernels and list-colorings

A kernel of a digraph D is an independent set $I \subseteq V(D)$, such that for every $x \in V(D) \setminus I$ there is $y \in S$, such that $\vec{xy}$.

Remark Not every digraph has a kernel.

Motivation The right-to-left orientation of the edges of a graph according to any ordering of its vertices has a kernel: the class of color 1 in the Greedy Coloring.

Definition A digraph is kernel-perfect if every induced subdigraph has a kernel.

Remark Every graph has an orientation that is kernel-perfect.

Let $f : V(G) \to \mathbb{N}$ be a function. A graph G is called f-choosable if a proper coloring can be chosen from any family of lists $\{L(x)\}_{x \in V(G)}$ provided $|L(x)| \geq f(x)$ for every $x \in V(G)$.

Lemma Let D be a kernel-perfect orientation of G. Then G is f-choosable with $f(x) = 1 + d_D^+(x)$.

Kernel-perfect orientation of $L(K_{n,n})$

Theorem (Galvin, 1995) $\chi'_l(K_{n,n}) = \chi'(K_{n,n})$.

Proof. Trivially, $n = \Delta(K_{n,n}) \leq \chi'(K_{n,n}) \leq \chi'_l(K_{n,n})$

Goal: construct kernel-perfect orientation D of $L(K_{n,n})$ such that $\Delta^+(D) = n - 1$ and then use Lemma to conclude that $L(K_{n,n})$ is f-choosable with $f \equiv \Delta^+(D) + 1 = n$.

Observation: In any kernel-perfect orientation D of $L(K_{n,n})$ the clique $D[\{vu : u \in N(v)\}]$ is transitively oriented, for every $v \in V(K_{n,n})$.

Claim 1. There is an orientation D of $L(K_{n,n})$ such that $\Delta^+(D) = n - 1$ and for every $v \in V(K_{n,n})$ the restriction of D to $\{vu : u \in N(v)\}$ is transitive.

Claim 2. Let D be an orientation of $L(K_{n,n})$ such that for every $v \in V(K_{n,n})$ the restriction of D to $\{vu : u \in N(v)\}$ is transitive. Then D is kernel perfect. $\square$

Orienting $L(K_{n,n})$

Proof of Claim 1.
$M = \{m_0, \ldots, m_{n-1}\}$ and $W = \{w_0, \ldots, w_{n-1}\}$

$E(K_{n,n}) = V(L(K_{n,n})) = \{m_i w_j : i, j \in [n]\}$

For $a \in \mathbb{N}$, let $r(a) \in \mathbb{N}$ be the residue of a modulo n. (That is, $r(a) \equiv a \pmod{n}$ and $0 \leq r(a) \leq n-1$)

Define: $m_i w_j \to m_{i'} w_j$ if $r(i+j) > r(i'+j)$

$m_i w_j \to m_i w_{j'}$ if $r(i+j) < r(i+j')$

Then $d^+(m_i w_j) = r(i+j) + n - 1 - r(i+j) = n-1$ for every $i, j \in [n]$

For fixed $w_j \in W$, incident edges are transitively oriented from the edge $m_{n-j-1} w_j$ (the source) towards the edge $w_{n-j} w_j$ (the sink), going around modulo n.

For fixed $m_i \in M$, incident edges are transitively oriented from the edge $m_i w_{n-i}$ (the source) towards the edge $m_i w_{n-i-1}$ (the sink), going around modulo n. □

Kernels through stable matchings

Interpret the transitive orientation on $D[\{vu : u \in N(v)\}]$ as the preferences of v about its neighbors.

A kernel of D then translates to a perfect matching $I \subseteq V(D) = E(K_{n,n})$, between M and W, such that for every edge $mw \in E(K_{n,n}) \setminus I$, at least one of m and w prefers its I-partner to the other.

Bonnie and Clyde is called an unstable pair if

- Bonnie and Clyde are currently not a couple,
- Bonnie prefers Clyde to her current partner, and
- Clyde prefers Bonnie to his current partner.

A perfect matching (of n women and n men) is called a stable matching if it yields no unstable pair.

Theorem. (Gale-Shapley, 1962) Let us be given for each of n men and n women arbitary preference rankings of the members of the opposite sex. Then there is a stable matching.

Nobel prize, 2012: Roth & Shapley "for the theory of stable allocations and the practice of market design."

Concluding kernel-perfectness

Proof of Claim 2.
Given an arbitrary subset $S \subseteq V(D)$, we define appropriate preference lists, such that for a corresponding stable matching I, $I \cap S$ is a kernel.

Man $m \in M$ prefers woman $w \in W$ to woman $w' \in W$ if

$$mw, mw' \in S \text{ and } mw \leftarrow mw' \text{ or}$$
$$mw \in S, mw' \notin S \text{ or}$$
$$mw, mw' \notin S \text{ and } mw \leftarrow mw'$$

This is a preference ranking of W for every $m \in M$, because $D[\{mw : w \in W\}]$ is transitive

Woman $w \in W$ prefers man $m \in M$ to man $m' \in M$ if

$$mw, m'w \in S \text{ and } mw \leftarrow m'w \text{ or}$$
$$mw \in S, m'w \notin S \text{ or}$$
$$mw \notin S, m'w \notin S \text{ and } mw \leftarrow m'w$$

This is a preference ranking of M for every $w \in W$, because D restricted to $\{mw : m \in M\}$ is transitive

There goes your kernel

Proposition. $I \cap S$ is a kernel for $D[S]$

Proof. I is a matching $\Rightarrow I \cap S$ is independent in D

Let $mw \in S \setminus (I \cap S)$ be arbitrary and let us find an out-neighbor in $I \cap S$.

Let $mw_m, m_w w \in I$ be the respective edges in the stable matching.

Since $mw \notin I$ is not an unstable pair for the stable matching I, either m prefers w_m to w, or w prefers m_w to m.

- If m prefers w_m to w, then $mw_m \in S$ (since $mw \in S$), and then $mw \to mw_m \in I \cap S$.
- If w prefers m_w to m, then $m_w w \in S$ (since $mw \in S$), and then $mw \to m_w w \in I \cap S$.

In both cases an out-neighbor in $I \cap S$ was found. $\square$

The proof of divorce-free society

Proposal Algorithm (Gale-Shapley, 1962)

Input. Preference ranking by each of n man and n woman.

Iteration.
Each man proposes to the woman highest on his list who has not previously rejected him.

IF each woman receives exactly one proposal, THEN
 stop and **report** the resulting matching as *stable*.

ELSE
 every woman receiving more than one proposal rejects all of them except the one highest on her list.
 Every woman receiving at least one proposal says "maybe" to the most attractive proposal she received.

Iterate.

Theorem. The Proposal Algorithm produces a stable matching.