Tibor Szabó
Michael Anastos

Discrete Maths I Review

Do date: Oct 15th at 4:15 PM and Oct 17th at 8:30 AM.

This sheet is intended to help you review some concepts from Discrete Mathematics I that might be useful for this semester's course. If you would like to review these topics together, come to the exercise class on the 15th of October or 17th of October, where you will get to discuss and work on these problems in small groups.

Exercise 1 Show that there does not exist a graph with degree sequence $(1, 6, 5, 4, 6, 2, 3)$.

Exercise 2 In a party there are $n \geq 3$ people. Every two people are either friends (acquaintances) or enemies (strangers). Show that there are either (i) 2 people with the same number of friends and a common friend or (ii) 2 people with the same number of enemies and an enemy in common.

Exercise 3

(i) Determine the chromatic and independence numbers of the cycle C_n.

(ii) Show that for any graph G, $\frac{v(G)}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$, where $\chi(G)$ and $\alpha(G)$ denote the chromatic and independence numbers respectively. Can you find examples where these bounds are tight?

(iii) Show that for any graph G, $\chi(G)\chi(\bar{G}) \geq n$, where $n = |V(G)|$ and $\bar{G}$ is the complement of G i.e. an edge is present in $\bar{G}$ iff it is not present in G.

Exercise 4 Let $k \in \mathbb{N}$ be a constant. For each pair of functions $f(n)$ and $g(n)$ from the functions given below, determine whether $f = o(g)$, $f = O(g)$, or $f = \Omega(g)$ (as n tends to infinity):

$$n^{\frac{1}{\log n}}, \binom{n}{k}, n^n, 2^{2^{\log^2 n}}, 2^{n^2}, n!, \log n.$$

Asymptotics and Estimates Let f, g be functions from $\mathbb{N}$ to $\mathbb{R}$. Then we use the following asymptotic notation:

- $f = o(g)$ if $\lim_{n\to\infty} f/g = 0$,
- $f = O(g)$ if there exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $|f(n)| \le Cg(n)$ for all $n \ge n_0$, and
- $f = \Omega(g)$ if $g = O(f)$.
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Asymptotically, Stirling's approximation is the most powerful thing we have:

$$n! = (1 + O(1/n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

But we usually use the following estimates. Using $(1 + 1/k)^k \le e$ for all $k \ge 1$, we can show that

$$\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n.$$

For binomial coefficients we have

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{en}{k}\right)^k.$$

The middle binomial coefficient can be estimated as follows.

$$\frac{2^{2k}}{2\sqrt{k}} \le \binom{2k}{k} \le \frac{2^{2k}}{\sqrt{2k}}.$$

Or, we can use Stirling's approximation to see the truth

$$\binom{2k}{k} = \frac{2^{2k}}{\sqrt{\pi k}}(1 + o(1))$$

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