Extremal Combinatorics Wi 2019–20
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Exercise Sheet 1

Due date: 14:15, 22th October[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Show that there exist a constant $c > 0$ and an integer n^* such that for $n \geq n^*$ every graph on n vertices has either (i) cn^3 triangles or (ii) cn^3 independent sets of size 3.

Remark: The above problem can be generalized to the following result: For every $t \in \mathbb{N}$ there exists a constant $c = c(t) > 0$ and an integer $n^* = n^*(t)$ such that for $n \geq n^*$ every graph on n vertices has either (i) cn^t cliques of size t or (ii) cn^t independent sets of size t.
Comment 1: This is optimal, modulo the constant factor c, since every graph on n vertices has $\binom{n}{t} = O(n^t)$ cliques of size t/independent sets of size t.
Comment 2: The above problem is a generalisation of the corresponding Ramsey question. One asks for many cliques/independent sets to appear instead of a single one.

Exercise 2 In this exercise you will prove the classic Erdős–Szekeres bound on the Ramsey numbers. Recall that $R(s,t)$ is the minimum integer n for which every `red`/`blue`-colouring of the edges of K_n contains a monochromatic K_s in `red` or a monochromatic K_t in `blue`.

(i) Show that for any $s, t \geq 2$, $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

(ii)* Prove that $R(s,t) \leq \binom{s+t-2}{s-1}$ for any $s, t \geq 2$.

(iii) Conclude that for the symmetric Ramsey number we have $R(s) = O(\frac{4^s}{\sqrt{s}})$.

Remark. This gives us an improvement by a factor of $\sqrt{s}$, over what we have proved in the lecture—which pales unfortunately, when compared to the exponential factor gap between the upper and lower bounds we know.

[1]Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 3

(*a*) Show that $R(3,4) \leq 10$.

(*a*) Improve (a) to $R(3,4) \leq 9$.

(*a*) Show that $R(3,4) > 8$.

Conclude that $R(3,4) = 9$.

Exercise 4* For the multicolour Ramsey numbers prove that

$$R_r(t_1, t_2, \ldots, t_r) \leq r^{1+\sum_{i=1}^{r}(t_i-1)}.$$

Hints to starred questions (read in reverse):
2(ii): ssulpt no noitcudni yrt.
4: 2=r htiw esac cirtemmys eht rof ssalc eht ni nevig foorp eht cimim