Extremal Combinatorics Wi 2019–20
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Exercise Sheet 10

Due date: 14:15, 14th January[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let $SF(\ell, k)$ be the smallest integer N, such that every family of N sets of size ℓ contains a sunflower with k petals.

(a) Show that $SF(2,3) = 7$.

(b) Show that for every even ℓ, we have $SF(\ell, 3) > \sqrt{6}^{\ell}$.

Exercise 2 Let UD^n be the unit distance graph in $\mathbb{R}^n$. Prove that $\chi(\mathrm{UD}^n) \leq 9^n$.

Exercise 3 Let X be a finite set and $\mathbb{F}$ a field. Let $M : X \times X \times X \mapsto \mathbb{F}$ be a 3-tensor. We say that M has tensor rank 1 if it can be written as $M(x,y,z) = f(x)g(y)h(z)$ for some functions $f, g, h : X \mapsto \mathbb{F}$. In general, the tensor rank of M is the least n for which there exist tensors $M_1, M_2, ..., M_n : X \times X \times X \mapsto \mathbb{F}$ of tensor rank 1 such that $M = \sum_{i=1}^{n} M_i$.

(a) Prove that the tensor rank of M is at most $|X|$ times its slice rank.

(b) Give an example of a 3-tensor $M : X \times X \times X \mapsto \mathbb{F}$ for which the slice rank of M is equal to 1 and the tensor rank of M is equal to $|X|$.

Exercise 4

Definition 1. Let G be an abelian group. A *3-colored sum-free set over G* is a collection $S \subseteq G^3$ of triples (x_i, y_i, z_i), $1 \leq i \leq L$, of elements of G such that, for all $i_1, i_2, i_3 \in [L]$, we have

$$x_{i_1} + y_{i_2} + z_{i_3} = 0 \text{ if and only if } i_1 = i_2 = i_3.$$

The size of a 3-colored sum-free set is the number L of triples it contains.

[1]Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

The goal of this exercise is to show the following theorem:

Theorem 1. *Let $m = p^\ell$ for some prime p and $\ell \geq 1$. Then, the size of any 3-colored sum-free set over $\mathbb{Z}_m^n$ is at most $\left(\frac{m}{e^{1/18}}\right)^n$*

We can split the proof into the following steps.

(a) Show that over $\mathbb{F}_p$ we have, for all $x, y, z \in \mathbb{Z}_m$,

$$\sum_{\substack{\delta_1,\delta_2,\delta_3 \in \{0,\ldots,m-1\} \\ \delta_1+\delta_2+\delta_3 \leq m-1}} (-1)^{\delta_1+\delta_2+\delta_3} \binom{x}{\delta_1}\binom{y}{\delta_2}\binom{z}{\delta_3} = \begin{cases} 1 & \text{if } x+y+z=0 \text{ in } \mathbb{Z}_m, \\ 0 & \text{otherwise.} \end{cases}$$

You may use the fact that if $0 \leq a \leq m-1$ and $z_1 \equiv z_2 \pmod{m}$, then $\binom{z_1}{a} \equiv \binom{z_2}{a} \pmod{p}$.

(b) Given a 3-colored sum-free set S over $\mathbb{Z}_m^n$ of size L, use the result from part (a) to construct a diagonal 3-tensor $M : S^3 \to \mathbb{F}_p$. By bounding its slice rank show that

$$L \leq 3\left(\frac{m}{e^{1/18}}\right)^n.$$

You may use Hoeffding's inequality: Let $\delta_1, \ldots, \delta_n$ be independent random variables taking values in $[a, b]$. Let $\delta = \sum_{i=1}^n \delta_i$. Then for $t \geq 0$

$$\mathbb{P}[\mathbb{E}(\delta) - \delta \geq t] \leq \exp\left(\frac{-2t^2}{n(a-b)^2}\right).$$

(c) Show that, given a 3-colored sum-free set over $\mathbb{Z}_m^n$ of size L and a positive integer k, we can construct a 3-colored sum-free set over $\mathbb{Z}_m^{nk}$ of size L^k. Deduce that $L \leq \left(\frac{m}{e^{1/18}}\right)^n$.

Hint for exercise 2: As a first step, show the existence of a maximal set $S \subseteq \mathbb{R}^n$ of points such that the distance between any two of them is at least $1/2$. Properly colour the infinite graph formed on the point set S by making two points adjacent if their distance is at most 2, using at most 9^n colours, and then use this colouring to obtain a proper colouring of the unit distance graph.

Hint for exercise 4a: Find another expression for the binomial coefficient $\binom{x++y+z}{a}$.