Extremal Combinatorics — Wi 2019–20

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Exercise Sheet 11

Due date: 14:15, 21st January[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Given two natural numbers k and n, the k-cascade representation of n is given by

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

where $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$. Prove that such a representation of n exists and is unique.

Exercise 2 Show that the skew version of the Bollobás set-pairs inequality is false in the non-uniform setting. More specifically, for each $n \in \mathbb{N}$, find a sequence of sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ such that

(1) $A_i \cap B_i = \emptyset$ for all i, and

(2) $A_i \cap B_j \neq \emptyset$ for all $i > j$, but

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \geq n + 1.$$

Exercise 3 Let $k \in \mathbb{N}$ be fixed. Let $\mathcal{M}$ be the set of all maximal k-uniform intersecting families over $[n]$.

(a) Show that $|\mathcal{M}| \leq 2^{n^k}$.

(b) Prove that for every $\mathcal{F} \in \mathcal{M}$ there exists a subset $\mathcal{S}_{\mathcal{F}} \subseteq \mathcal{F}$ of size at most $\frac{1}{2}\binom{2k}{k}$ such that $\mathcal{S}_{\mathcal{F}} \nsubseteq \mathcal{F}'$ for every $\mathcal{F}' \in \mathcal{M}, \mathcal{F}' \neq \mathcal{F}$.

(c) Deduce that $|\mathcal{M}| \leq 2^{k2^{2k}\log_2 n}$.

(d) (Bonus) Prove that $\binom{n}{2k-2}(n-2k+2)^{\frac{1}{2}\binom{2k-2}{k-1}} \leq |\mathcal{M}|$.

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 4 Let n be odd. Sperner's Theorem states that if $\mathcal{F} \subseteq 2^{[n]}$ is a family of subsets such that for every $A, B \in \mathbb{F}$ we have $A \not\subseteq B$ then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Prove that the families of sets that achieve the upper bound in the above theorem are exactly the ones corresponding to the middle levels i.e. the sets $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$.

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Hint to Exercise 3: Let S_F be a minimal set that satisfies $\{A \in \binom{[n]}{k} : B \cap A \neq \emptyset \text{ for all } B \in S_F\} = F$

Hint to Exercise 4: For an intersecting set F consider $F' = \{A^c : A \in F\}$.