Extremal Combinatorics Wi 2019–20
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Exercise Sheet 11

Due date: 14:15, 28th January[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 For $x \in \mathbb{R}$, and $k \in \mathbb{N}$, define

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

for all $x \geq k$. Since this is an increasing (continuous) function in this domain, we see that for every integer m there exists a unique x for which $m = \binom{x}{k}$.

Prove the following weaker version of Kruskal-Katona theorem that is due to Lovász (you may use any claims/lemmas proved in the lecture notes):

Theorem 1 (Lovász)**.** *Let $\mathcal{F}$ be a k-uniform set-family of size $m = \binom{x}{k}$, for some real number $x \geq k$. Then*

$$|\partial F| \geq \binom{x}{k-1}.$$

Remark: While Kruskal-Katona gives a sharp bound on the shadow size, in practice it is usually better to use its weaker, but computationally friendlier, version stated above.

Exercise 2 Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$, define its ℓ-shadow to be

$$\partial_\ell(\mathcal{F}) = \{E \in \binom{[n]}{\ell} : E \subset F \text{ for some } F \in \mathcal{F}\}.$$

(1) For $0 \leq \ell < k$ and $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ for $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$, determine the smallest possible size of the ℓ-shadow of a set family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size m.

(2) Derive the Erdős-Ko-Rado theorem from (1): if $n \geq 2k$, then the largest intersecting family in $\binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 3 Prove the following slightly stronger version of the 2-dimensional case of Sperner's Lemma:

Let T be a triangle, and let $\mathcal{T}$ be a triangulation of T. Let C be a Sperner colouring of $\mathcal{T}$ that colors the vertices of T by colors $1, 2$ and 3 in clockwise order. Show that there is a triangle $T' \in \mathcal{T}$ with vertices of color 1, 2 and 3 in clockwise order.

Exercise 4 In this exercise we are going to prove the following: Let $T \subset \mathbb{R}^2$ be a triangle and $f : T \mapsto T$ be a continuous function. Then there exists a point $x \in T$ such that $f(x) = x$.

W.l.o.g we may assume assume that T is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ (if not we may define a continuous bijection from our initial to our desired triangle) that is T is the set of points in $\mathbb{R}^3$ with nonnegative coordinates whose coordinates sum to 1. Show that for any continuous function $f : T \mapsto T$ Sperner's Lemma implies that there exists $x \in T$ such that $f(x) = x$.

Bonus: Show that the 2-dimensional version of Sperner's Lemma implies that for any triangle $T \subset \mathbb{R}^2$ and any continuous function $f : T \mapsto T$ there exists a point $x \in T$ such that $f(x) = x$.

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Hint to Exercise 2: For an intersecting set F consider $F' = \{A^c : A \in F\}$.