Extremal Combinatorics Wi 2019–20

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Supplementary Exercise Sheet[1]

Due date: NA

Exercise 1 Consider the two statements below.

(BU) For any continuous map $f : S^d \to \mathbb{R}^d$, there is some $x \in S^d$ such that $f(x) = f(-x)$.

(SC) If $S^d = U_0 \cup U_1 \cup \ldots \cup U_d$, where for each $1 \leq i \leq d$, U_i is either open or closed, then there is some $0 \leq j \leq d$ such that U_j contains a pair of antipodal points $\{x, -x\}$.

In lecture we showed (BU) $\Rightarrow$ (SC). Show that they are in fact equivalent by proving (SC) $\Rightarrow$ (BU).

Exercise 2 A function $f : S \mapsto S'$ is called odd if for every $x \in S$, $f(-x) = -f(x)$. A retraction from S^d to S^{d-1} is a function $f : S^d \mapsto S^{d-1}$.

Consider the three statements below.

(BU) For any continuous map $f : S^d \to \mathbb{R}^d$, there is some $x \in S^d$ such that $f(x) = f(-x)$.

(E1) For any continuous odd map $f : S^d \to \mathbb{R}^d$, there is some $x \in S^d$ such that $f(x) = 0$.

(E2) No retraction from S^d to S^{d-1} is continuous and odd.

Show the following:

(a) (BU), (E1) and (E2) are equivalent.

(b) Show that (BU) implies Brouwer fixed-point theorem.

Exercise 3 This exercise consists of the following two parts.

(a) Let $K \subset \mathbb{R}^2$ be a compact set. Show that there exist two hyperplanes h_1 and h_2 that together divide K into four parts of equal area.

(b) Recall that we say a hyperplane h bisects a finite point set A if $|h^+ \cap A|, |h^- \cap A| \leq \lfloor \frac{1}{2}|A| \rfloor$. One might think that it would instead make sense to count any point on the hyperplane with weight $\frac{1}{2}$ in each half-space. That is, you could say h bisects A if $|h^+ \cap A| + \frac{1}{2}|h \cap A| = \frac{1}{2}|A| = |h^- \cap A| + \frac{1}{2}|h \cap A|$. Show that, with this definition of bisection, there are finite sets of points A_1 and A_2 in $\mathbb{R}^2$ that cannot be simultaneously bisected by a line.

[1]The purpose of this sheet is to practise concepts presented during the last weeks of the course

Exercise 4 Prove that if A_1 and A_2 are disjoint sets of n points in $\mathbb{R}^2$ such that $A_1 \cup A_2$ is in general position, there is a perfect matching from A_1 to A_2 such that the straight line segments between the matched pairs are pairwise disjoint.