Tibor Szabó
Michael Anastos

Supplementary Exercise Sheet 2[1]

Due date: NA

Exercise 1 Let R be a necklace that contains s different stones each appearing a multiple of k times. Let $cuts(s,k)$ be the minimum number $n \in \mathbb{N}$ that one can cut R into and then partition the resultant pieces into k sets such that for $i \in [s]$ $stone_i$ appears the same number of times in each of the k sets. In class we showed that $cuts(s,2) = s$. Prove the following statement:

If $cuts(s,k_1) = s(n_1 - 1)$ and $cuts(s,k_2) = s(k_2 - 1)$ then, $cuts(s,k_1k_2) = s(k_1k_2 - 1)$.

Exercise 2 Let $C_1, C_2, ..., C_k$ be a partition of the interval $[0, kn]$ into k measurable sets such that the Lebesgue measure of C_i, denoted by $\mu(C_i)$, equals 1 for $i \in [k]$. Prove that one can partition the interval $[0, kn]$ into k sub-intervals $I_1, ..., I_k$ and then partition the subintervals into two sets $\mathcal{I}_1, \mathcal{I}_2$ such that for $i \in [k]$ the following holds:

$$\sum_{I \in \mathcal{I}_1} \mu(I \cap C_i) = \sum_{I \in \mathcal{I}_2} \mu(I \cap C_i) = 1/2.$$

Exercise 3 For $d \geq 2$ and a set of points P in $\mathbb{R}^d$ let U_P be the unit distance graph with vertex set P. Let $e_P = |E(U_p)|$ and define $u(d,n)$ by

$$u(d,n) := \max_{P \subset R^d : |P| = n} e_P.$$

Show that for $d \geq 4$,

$$u(d,n) \geq \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right)n^2 + o(n^2).$$

Exercise 4 Let G be an n-vertex unit graph in $\mathbb{R}^n$.

(a) Show that G does not span the complete $(\lfloor d/2 \rfloor + 1)$-partite graph $K_{3,3,...,3}$

(b) Deduce that $u(d,n) \leq \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right)n^2 + o(n^2)$.

[1]The purpose of this sheet is to practise concepts presented during the last weeks of the course