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Exercise Sheet 2

Due date: 14:15, 29th October[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Deduce the finite Ramsey theorem from the infinite case. That is, using the fact that every `red/blue` colouring of $\binom{\mathbb{N}}{2}$ contains an infinite monochromatic clique, show that for every positive integer $t \geq 2$, there exists a finite n such that every `red/blue` colouring of $E(K_n)$ contains a monochromatic K_t.

Exercise 2 Let $\bar{R}_r(t)$ be the upper bound on the multicolored Ramsey number $R_r(t, t, ..., t)$, $t \geq 3$, that can be obtained using the "colorblind" argument that we used in class (see page 4 of the lecture notes of Week 1). In the class we showed that $\bar{R}_r(t)$ can be as big as a tower of 2's of height r. In this exercise we will show that this is far from the true and that actually $\bar{R}_r(t)$ is a double exponential[2].

At class we repeatedly used the inequality $R(k, l) \leq \binom{k+l-2}{k-1} \leq 2^{k+l}$. Show that for $l \leq k$ actually, $R(k, \ell) \leq (2k)^{\ell}$. Use this bound to show that $\bar{R}_r(t)$ is upper bounded by a double exponential function in r, t.

Comment : This is no far from the true as someone, using the result given in Exercise 4, can show that $\bar{R}_r(t)$ is a double exponential function in r, t.

Exercise 3 Prove the following bounds on the hypergraph Ramsey numbers.

(i) Show that for $k \geq 2$ and $s, t \geq k + 1$,

$$R^{(k)}(s, t) \leq R^{(k-1)}(R^{(k)}(s - 1, t), R^{(k)}(s, t - 1)) + 1.$$

Use the above inequality to show that for every $k \geq 2$ and every $s, t \geq k + 1$ the Ramsey number $R^{(k)}(s, t)$ is finite.

(ii) Deduce the bound $R^{(3)}(4, 4) \leq 19$.

[1]Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

[2]double exponential: we say that $\bar{R}_r(t)$ is double exponential in r, t if it can be bounded by a function of the form $e^{e^{g(r,t)}}$ where $g(r, t)$ is a polynomial in r, t

Exercise 4

(a) By considering a random coloring, show that for every $0 \leq p \leq 1$ and every integer n, $R(k, \ell)$ satisfies the following inequality:

$$R(k, \ell) \geq n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}. \tag{1}$$

(b) Let $4 \leq \ell \leq k \leq n$. Show that if n satisfies

$$1 \leq \frac{k^{\frac{2}{k-1}} n^{-\frac{2}{k}}}{4^{\frac{2}{k(k-1)}} e^{\frac{2}{k-1}}} + \frac{\ell^{\frac{2}{\ell-1}} n^{-\frac{2}{\ell}}}{4^{\frac{2}{\ell(\ell-1)}} e^{\frac{2}{\ell-1}}} \tag{2}$$

then, $R(k, \ell) \geq n/2$.

(c) Let $4 \leq \ell \leq k \leq n$. Show that if n satisfies

$$1 \leq n^{-\frac{2}{k}} + n^{-\frac{2}{\ell}} \tag{3}$$

then, $R(k, \ell) \geq n/2$.

(d) Using the inequality $1 - x \leq e^{-x}, x \in \mathbb{R}$ show that

$$R(k, \ell) \geq 0.5 \left(\frac{k}{\ell \log k} \right)^{\ell/2}.$$