Exercise Sheet 4

Due date: 14:15, 12th November[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise, you will complete the details of the Cherkashin–Kozik lower bound on the size of the smallest non-two-colorable k-graph. Let H be a k-graph with m edges. Recall that for each vertex $v \in V(H)$, we independently sample $x_v \sim U([0,1])$, a uniformly random number in $[0,1]$. We then order the vertices in increasing order of x_v, and run the greedy algorithm in that order. That is, we color a vertex blue unless it is the last vertex in an all-blue edge, in which case it is colored red.

(a) Consider the following events, where $\delta \in (0,1)$.

  (i) $\mathcal{L}_e = \left\{\forall v \in e : x_v < \frac{1}{2}(1-\delta)\right\}$ for some edge $e \in E(H)$.

  (ii) $\mathcal{R}_f = \left\{\forall v \in f : x_v > \frac{1}{2}(1+\delta)\right\}$ for some edge $f \in E(H)$.

  (iii) $\mathcal{E}_{e,f} = \{|e \cap f| = 1,$ the last vertex v of e is the first vertex of f, and $x_v \in \left[\frac{1}{2}(1-\delta), \frac{1}{2}(1+\delta)\right]\}$ for two edges $e, f \in E(H)$.

  Show that $\mathbb{P}(\mathcal{L}_e) = \mathbb{P}(\mathcal{R}_f) = (1-\delta)^k 2^{-k}$ and $\mathbb{P}(\mathcal{E}_{e,f}) \leq \delta 2^{2-2k}$.

(b) Let $m = \beta 2^{k-1}$. Show that if $\beta(1-\delta)^k + \beta^2\delta < 1$, then H is two-colorable.

(c) By choosing β and δ appropriately, show that there is some positive constant $c > 0$ such that $m_B(k) \geq c\left(\frac{k}{\ln k}\right)^{\frac{1}{2}} 2^k$.

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 2 In this exercise we will prove an upper bound on the happy ending number HE(t) without relying on the hypergraph Ramsey numbers. Recall that HE(t) is the smallest number of points in $\mathbb{R}^2$ in general position which ensure that we always have t points from them, forming a convex subset.

A sequence of consecutive line segments in $\mathbb{R}^2$ is called a **Cap** if their slopes are monotonically decreasing, and a **Cup** if their slopes are monotonically increasing (see the figure below). Let $f(s,t)$ denote the smallest number for which any collection of $f(s,t)$ points in general position either contains a **Cap** of length s or a **Cup** of length t.

(a) Prove that $f(s,3) = s$ and $f(t,3) = t$ for all $s,t \geq 3$.

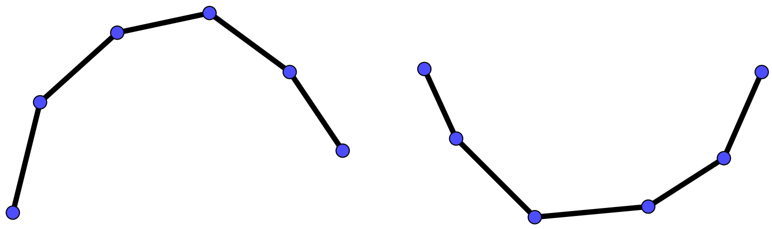
(b) Prove that $f(s,t) \leq f(s-1,t) + f(s,t-1) + 1$ for all $s,t \geq 4$. (Hint: What if there are at least $f(s,t-1)$ points that are the left most points of the **Caps** of length $s-1$)

(c) Deduce that

$$f(s,t) \leq \binom{s+t-2}{s-2} + 1.$$

(d) Show that the happy ending number can be bounded from above as follows:

$$\mathrm{HE}(t) \leq \binom{2t-4}{t-2} + 1.$$



Cap **Cup**

Remark: The bound that we have obtained is much better than any of the upper bounds obtained using the hypergraph Ramsey numbers. The lower bound of Erdős and Szekeres was $2^{t-2}+1$ and this is what they conjectured to be the truth. After essentially no significant progress for the last 80 years, the upper bound was recently improved to $2^{t(1+o(1))}$ by Andrew Suk.

Exercise 3

(i) Let H be a 3-uniform Hypergraph on $n \geq 5$ points in which each pair of points occurs in the same (positive) number of edges. Prove that H is not 2-colorable.

(ii) Find a 3-uniform Hypergraph on 7 vertices that is not 2-colorable.

Exercise 4 Let $\epsilon > 0$ and $d > 0$ be fixed. Prove that for $k \geq 2$, if $d \geq (1+\epsilon)2k(\log k + 1)$ then,

$$\lim_{n\to\infty} \mathbb{P}\left[G\left(n, \frac{d}{n}\right) \text{ is } k\text{-colorable }\right] \to 0.$$

Remark: With probability tending to 1 the chromatic number of $G\big(n, \frac{d}{n}\big)$ is $(1+o_d(1))\frac{d}{2\log d}$.