Extremal Combinatorics Wi 2019–20
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Exercise Sheet 5

Due date: 14:15, 19th November[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let G be the graph on vertex set $[4]$ with edges $12, 13, 14, 23$ (triangle with a tail). Determine $ex(n, G)$ for every n.

Exercise 2 In this exercise you will determine the asymptotic behaviour of $\mathrm{ex}(n, C_4)$. Let $\mathbb{F}_q$ be the finite field of order q, where q is a prime power [2]. Let $\mathcal{P} = \{(x, y) : x, y \in \mathbb{F}_q\}$ be the set of points in the plane $\mathbb{F}_q^2$ and let

$$\mathcal{L} = \{\{(x, y) : ax + by = c\} : a, b, c \in \mathbb{F}_q \text{ and at least one of } a, b \text{ is non-zero}\}$$

be the set of lines. Define a bipartite graph G_q with the two parts as $\mathcal{P}$ and $\mathcal{L}$ and making an edge between a point $P = (x, y)$ and a line ℓ if P lies on ℓ, i.e., its coordinates (x, y) satisfy the equation of ℓ.

(1) Show that the graph G_q has $2q^2 + q$ vertices and $q^3 + q^2$ edges.

(2) Show that the graph G_q does not contain any C_4's, and thus deduce that there exists a constant C such that for infinitely many n, we have $\mathrm{ex}(n, C_4) \geq Cn^{3/2}$.

(3) Conclude that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ [3].

Exercise 3 In this exercise you will prove the Kővari-Sós-Turán theorem.

Let G be an n-vertex $K_{s,t}$-free graph. By counting the copies of $K_{1,s}$ in G in two ways show that $\mathrm{ex}(n, K_{s,t}) \leq c_t n^{2-1/s}$, for some constant c_t that depends on t.

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

[2] for example, when q is equal to a prime p you can take this to be the residues modulo p

[3] Hint: "there is always a prime between n and $2n$." (bonus: find out who said this!)

Exercise 4 In this exercise you will show that a random bipartite graph is ε-regular with high probability, thus validating our intuitive understanding that ε-regular pairs are "random looking".

Fix some $\varepsilon > 0$ and some probability $p \in (0, 1)$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the two parts of the random bipartite graph $G(n, n, p)$ with bipartition $V(G) = A \cup B$, i.e. for all $1 \leq i, j \leq n$, the edge $\{a_i, b_j\}$ belongs to our random graph $G = G(n, n, p)$ with probability p, independently of all other edges. Show that

$$\mathbb{P}\left[(A, B) \text{ is } \varepsilon\text{-regular}\right] \to 1 \quad \text{as} \quad n \to \infty.$$

You may use the Chernoff bound, which in particular implies that if a random variable X is distributed as a *Binomial*(N, p) random variable then,

$$\mathbb{P}\left(|X - Np| \geq t\right) \leq 2e^{-2t^2/N}.$$

Remark: The Central limit theorem implies that if X is distributed as a *Binomial*(N, p) random variable then $(X - Np)/\sqrt{N}$ is distributed as a $Normal(0, p(1-p))$ random variable. Hence, X is concentrated in an interval of size $\Theta(\sqrt{N})$ around Np and the probability of X being far away from Np, say by $Np \log N$ tends to zero as N tends to infinity. The Chernoff bound gives you a bound on how "fast" the later probability tends to zero. Namely it implies that the probability of X being outside the interval of size $t = t'\sqrt{N}$ around Np is bounded by $2e^{-2(t')^2}$.