Extremal Combinatorics Wi 2019–20
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Exercise Sheet 6

Due date: 14:15, 26th November[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let G be a graph on n vertices which has no cycles of length less than g.

(a) If G has average degree $\bar{d}$ then, it has a subgraph H of minimum degree $\bar{d}/2$.

(b) If the degree of each vertex in G is at least d, then prove that the number of vertices in G is $\Omega(d^k)$ where $k = \lfloor \frac{g-1}{2} \rfloor$.

(c) For a family of graphs $\mathcal{F} = \{H_i : i \in I\}$ we let $ex(n, \mathcal{F})$ be the maximum number $m \in \mathbb{N}$ such that there exists a graph G on n vertices with m edges that does not contain as a subgraph any of the elements of $\mathcal{F}$. Let $\mathcal{F}_g$ be the set of cycles of size at most g. Prove that $ex(n, \mathcal{F}_g) = O(n^{1+1/k})$ where $k = \lfloor \frac{g-1}{2} \rfloor$.

Exercise 2 The TV remote of George requires two working batteries to function. Opening the drawer in which he keeps the batteries, he finds eight. He remembers that four of them work and four of them do not, but there is no way of telling them apart without testing them in the remote. How quickly can George guarantee to find two working batteries for his remote? (i.e. what is the minimum number of tests that will suffice even in the worst case?)

Exercise 3

(i) Let $G = (V, E)$ be a graph, and let $A, B \subset V$ be two disjoint non-empty sets of vertices, of sizes a and b respectively. Show that in order to check whether or not the pair (A, B) is ϵ-regular, it suffices to check if $|d(X, Y) - d(A, B)| \leq \epsilon$ for every pair (X, Y) with $X \subset A$ of size precisely $\lceil \epsilon a \rceil$ and $Y \subset B$ of size precisely $\lceil \varepsilon b \rceil$.

(ii) Let $\epsilon > 0$. Show that there exists $n_\epsilon \in \mathbb{N}$ such that for every $n > n_\epsilon$ the following statement holds: if $G = (V, E)$ is a graph on n vertices with the property that every edge of G is contained in precisely one triangle, then $|E| \leq \epsilon n^2$.

[1]Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 4 Show that for every $\eta \in (0,1)$ there exist $\gamma = \gamma(\eta)$ and $\delta = \delta(\eta) > 0$ with the following property.

For every graph G and every pairwise disjoint subset $V_1, \ldots, V_4$ of $V(G)$ if

(i) V_i, V_j is γ-regular for any $1 \leq i < j \leq k$,

(ii) $\mathrm{d}(V_i, V_j) \geq \eta$, if $\{i,j\} \in \{(1,2),(2,3),(3,4),(4,1)\}$ and $\mathrm{d}(V_i, V_j) \leq 1-\eta$, otherwise,

then, there exists at least $\delta \prod_{i=1}^{4} |V_i|$ induced copies of C_4.