Extremal Combinatorics Wi 2019–20
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Exercise Sheet 7

Due date: 14:15, 3rd December[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 [2] Let $k \in \mathbb{N}$. Show that whenever $\mathbb{N}^2$ is k-colored there exists $\mathbf{z} \in \mathbb{N}^2$ and $d_1, d_2 > 0$ such that the set

$$\left\{\mathbf{z} + ad_1\overrightarrow{e_1} + bd_2\overrightarrow{e_2} : a, b \in \{0,1,2\}\right\}$$

is monochromatic.

Exercise 2 Construct a coloring of the positive integers with finitely many colors such that there is no monochromatic solution to the equation $x + y = 3z$.

Exercise 3 A *corner* in $[n]^2$ is a triple of points of the form $\{(x,y),(x+h,y),(x,y+h)\}$ for some positive x and y and some non-zero h.[3] Show that for every $\varepsilon > 0$, if n is sufficiently large, then any $A \subset [n]^2$ with $|A| \geq \varepsilon n^2$ must contain a corner.

Exercise 4 For a pair of points $a, b \in \mathbb{R}^2$ define its midpoint, denoted by $m(a,b)$, to be the point $(a+b)/2$. Let $\epsilon > 0$ be fixed. Show that for every n there exists a set $S \subset \mathbb{R}^2$ of n points, no 3 collinear, such that

$$|\{m(a,b) : a, b \in S\}| \leq C(\varepsilon) n^{1+\epsilon}.$$

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

[2] $\overrightarrow{e_1} = (1,0)$ and $\overrightarrow{e_2} = (0,1)$.

[3] In other words, a corner is an axis-aligned isosceles right triangle. Also note that here h may be negative.

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Hint to Exercise 2: thehintstartsnowp-1colorforsomesufficientlylargeprimep.