Extremal Combinatorics Wi 2019–20
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Exercise Sheet 8

Due date: 14:15, 10th December[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Prove or disprove the following statement: Whenever the natural numbers $\mathbb{N}$ are two-coloured there exist an *infinite* monochromatic arithmetic progression $(a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \ldots)$.

Exercise 2 Let $a_1, \ldots, a_n \in \mathbb{Z}/\{0\}$ be constants such that $\sum_{i\in I} a_i = 0$ for some non-empty $I \subseteq [n]$. Show that whenever the natural numbers $\mathbb{N}$ are two-coloured the equation $a_1x_1 + \cdots + a_nx_n = 0$ has a monochromatic solution $(x_1, \ldots, x_n)$. You may use the following steps:

(a) Show that any two-coloring of the integers contains a monochromatic solution of the equation $x + 2y = z$.

(b) Show that for any $\alpha \in \mathbb{Q}$ any two-coloring of the integers contains a monochromatic solution of the equation $x + \alpha y = z$.

(c) Deduce the general case.

Remark It is also true that if there does not exist a nonempty $I \subseteq [n]$ such that $\sum_{i\in I} a_i = 0$ then there exists a coloring that yields no monochromatic coloring to the equation $a_1x_1 + \cdots + a_nx_n = 0$. Such a coloring may be constructed analogously to the coloring given in the solution of Exercise 2/Sheet 7.

Exercise 3 Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of subsets of $[n]$, such that for every $i \neq j$, $F_i \not\subseteq F_j$, $F_i \cap F_j \neq \emptyset$, $F_i \cup F_j \neq [n]$. Prove that

$$m \leq \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

Exercise 4 For a pair of k-uniform set families $\mathcal{F}_1, \mathcal{F}_2$ on disjoint ground sets X_1 and X_2 respectively let $\mathcal{F}_1 \times \mathcal{F}_2$ be the $2k$-uniform set family defined by

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{f_1 \cup f_2 : f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\}.$$

(a) Let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n$ be k-uniform set families, supported on pairwise disjoint ground sets, such that none of them contains a sunflower of size ℓ. Show that $\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ is a kn-uniform set family that does not contain a sunflower of size ℓ.

(b) For every even k contruct a k-uniform family of size $(\sqrt{5})^k$ with no sunflower of size 3.

Hint to Exercise 2a: TrytomakeuseofvanderWaerdenstheoreminyourproof.