Extremal Combinatorics Wi 2019–20
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Exercise Sheet 9

Due date: 14:15, 17th December[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let $\mathcal{F}$ be a family of sets in $2^{[n]}$ that does not contain a chain of size 3 [2]. Show that

$$\mathcal{F} \leq \binom{n}{\lceil n/2 \rceil} + \binom{n}{\lceil n/2 \rceil - 1}.$$

Exercise 2 For two vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field, let $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$ be the standard inner product [3]. For a set $S \subseteq \mathbb{F}^n$, let $S^\perp =: \{\mathbf{v} \in \mathbb{F}^n : \mathbf{v} \cdot \mathbf{u} = 0 \ \forall \mathbf{u} \in S\}$.

(1) Prove that if U is a (vector) subspace of $\mathbb{F}^n$, then $U^\perp$ is also a subspace and $\dim U + \dim U^\perp = n$.

Recall that in an Eventown we have distinct subsets of $[n]$ (clubs from a town of n people) such that every subset has even cardinality and every two distinct subsets share an even number of elements.

(2) Prove that in an *Eventown* the number of subsets is at most $2^{\lfloor n/2 \rfloor}$.

Exercise 3

(a) Let $\mathcal{F}_1, \mathcal{F}_2$ be two families of sets in $2^{[n]}$ such that $|f_1 \cap f_2|$ is even for every $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$. Show that

$$|\mathcal{F}_1||\mathcal{F}_2| \leq 2^n.$$

(b) Let $\mathcal{F}_1, \mathcal{F}_2$ be two families of sets in $2^{[n]}$ such that $|f_1 \cap f_2|$ is odd for every $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$. Show that

$$|\mathcal{F}_1||\mathcal{F}_2| \leq 2^{n-1}.$$

[1] Please submit the exercise sheet before 14:15 on Tuesday. You can submit it in the tutor box of Michael Anastos (box number B8, in front of lecture hall 001, Arnimallee 3-5) or at the beginning of the exercise class on Tuesday or electronically at manastos@zedat.fu-berlin.de

[2] There do not exist $A, B, C \in \mathcal{F}$ such that $A \subset B \subset C$

[3] it's more standard to call this a symmetric bilinear form and reserve the term "inner product" for the case when $\mathbb{F}$ is equal to $\mathbb{R}$ or $\mathbb{C}$.

Exercise 4 (Equiangular Lines) A set S of lines through origin in the n-dimensional Euclidean space $\mathbb{R}^n$ is called **equiangular** if every pair of lines in S has the same angle between them. In this exercise we will prove an upper bound on the number of equiangular lines in $\mathbb{R}^n$ using the linear algebra method.

(1) Give an example of such a set S in $\mathbb{R}^2$ with $|S| = 3$. Can we construct a larger equiangular set S in $\mathbb{R}^2$?

(2) Let $S = \{L_1, \ldots, L_k\}$ be a set of equiangular lines in $\mathbb{R}^n$ and let $\alpha \neq 1$ be the cosine of the common angle between these lines. For each i, let $\mathbf{u}^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})$ be an arbitrary unit vector on L_i. Use these $\mathbf{u}^{(i)}$'s to construct polynomial functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in \{1, \ldots, k\}$, for which we have

$$f_j(\mathbf{u}^{(i)}) = \begin{cases} 1 - \alpha^2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

and show that $f_1, \ldots, f_k$ are linearly independnet.

(3) By choosing f_i's appropriately, prove that $|S| \leq n(n+1)/2$.

Remark: This bound was proved by Gerzon in early 70's and the proof above was given by Koornwinder in 1976. This is the first instance of using a vector space of polynomials/functions in the linear algebra method in combinatorics. Moreover, it can also be seen as one of the first instances of the so-called polynomial method in combinatorics.