Freie Universität Berlin
Institut für Mathematik
Extremal Combinatorics

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Exercise	1.	2.	3.	4.	5.	6.
Points						

PRACTICE EXAM

You may use this practice exam to test your knowledge of the material covered in the course so far. It will not form part of your grade in any way, but if you would like to receive feedback on your solutions, please submit them for grading. We recommend that you submit them by the 7th of January 2020 as it will be discussed during the exercises classes on Janury 7th and on January 9th. In addition we will release a new exercise sheet on January 8th. However we expect to grade the practice exams during the second week of classes, hence any exam submitted by 09:00 on January 13th will be graded. We would recommend that you take this exam under exam conditions three hours, no notes.

Instructions Solve all the questions. Each question is worth 10 points. Show all your work and state precisely the theorems you are using from the lecture. No notes are allowed. The time limit is 3 hours.

Notation: $\mathbb{N} = \{1, 2, 3, ...\}$.

Question 1 [10 points]

(a) For $t_1, t_2, t_3 \in \mathbb{N}$ define the Ramsey number $R(t_1, t_2, t_3)$.

(b) For $k \in \mathbb{N}$ prove that

$$R(k,k,k) \geq (1 - o_k(1))\frac{k}{3e}\sqrt{3}^k.$$

Question 2 [10 points]

(a) Let H be a fixed given graph H and $n \in \mathbb{N}$. Define $ex(n, H)$.

(b) For $1 \leq t \leq s$ prove that $ex(n, K_{s,t}) \leq c_{s,t} n^{2-1/s}$ for some explicit constant $c_{s,t}$ that may depend on s, t.

Question 3 [10 points]

(a) State the definition of ϵ-regularity.

(b) Let $A, B, C \subseteq V(G)$ be pairwise disjoint sets in some graph G such that the pairs $\{A, B\}, \{B, C\}, \{A, C\}$ are all ϵ-regular with densities $d(A, B), d(A, C), d(B, C) \geq 2\epsilon$. Show that the number of triangles in G is at least

$$(1 - 2\epsilon)(d(A, B) - \epsilon)(d(A, C) - \epsilon)(d(B, C) - \epsilon)|A| \cdot |B| \cdot |C|.$$

Question 4 [10 points]

Show that whenever $\mathbb{N}$ is 2-colored there exists a monochromatic solution to the equation

$$x + 2.5y = z.$$

Question 5 [10 points]

Let $\mathcal{F}$ be a family of subsets of $[n]$ such that,

(a) $|F| \neq 0 \mod 6$, for $F \in \mathcal{F}$.

(b) $|F_i \cap F_j| = 0 \mod 6$, for every pair of distinct $F_i, F_j \in \mathcal{F}$.

Show that $|\mathcal{F}| \leq 2n$.

Question 6 [10 points]

A set S of points in $\mathbb{R}^n$ is called a **two-distance set** if the distance between pairwise distinct points of S takes only two distinct values. Let $m(n)$ be the maximum size of a two-distance set in $\mathbb{R}^n$.

(a) Prove that $m(n) \geq n(n-1)/2$.

(b) Prove that there exists a constant $C > 0$ such that $m(n) \leq O(n^C)$.

(c) (Not part of the exam) Show that C in part (b) can be taken to be equal to 2.

Solution suggestions to the practice exam

Question 1

(a)For $t_1, t_2, t_3 \in \mathbb{N}$ the Ramsey number $R(t_1, t_2, t_3)$ is the minimum integer n such that in every colouring of the edges of K_n with colors c_1, c_2, c_3 there exists $i \in [3]$ such that K_n contains either a monochromatic K_{t_i} in color c_i.

(b) Let $n \in \mathbb{N}$. Color the edges of K_n independently with color blue, red or green each with probability $\frac{1}{3}$. For $S \in \binom{[n]}{k}$ let X_S be the indicator of the event "S spans a monochromatic clique" and let $X = \sum_{S \in \binom{[n]}{k}} X_S$. Then,

$$\mathbb{E}(\text{number of monochromatic } k\text{-cliques induced by the random coloring})$$
$$= \mathbb{E}(X) = \mathbb{E}\left(\sum_{S \in \binom{[n]}{k}} X_S\right) = \sum_{S \in \binom{[n]}{k}} \mathbb{E}(X_S) = \binom{n}{k} \cdot 3\left(\frac{1}{3}\right)^{\binom{k}{2}}$$

Therefore there exists a colouring c of K_n such that the number of monochromatic K_k is at most $\binom{n}{k} \cdot 3\left(\frac{1}{3}\right)^{\binom{k}{2}}$. Fix such a colouring c and delete one vertex from each monochromatic K_k. This gives a 3-coloring on at least $n - \binom{n}{k} \cdot 3\left(\frac{1}{3}\right)^{\binom{k}{2}}$ vertices without any monochromatic K_t. Hence

$$R(k,k,k) \geq n - \binom{n}{k} \cdot 3\left(\frac{1}{3}\right)^{\binom{k}{2}} \geq n - \left(\frac{en}{k}\right)^k \cdot 3\left(\frac{1}{3}\right)^{k(k-1)/2} = n - 3\left(\frac{en}{k3^{k-1/2}}\right)^k.$$

The above relation is true for every $n \in \mathbb{N}$. Substituting $n = \frac{k}{e}\sqrt{3}^k$ gives,

$$R(k,k,k) \geq \frac{k}{e}\sqrt{3}^k - 3\left(\frac{e\frac{k}{e}\sqrt{3}^k}{k3^{k-1/2}}\right)^k = \frac{k}{e}\sqrt{3}^k - 3(3^{1/2})^k = (1 - o_k(1))\frac{k}{e}\sqrt{3}^k.$$

□

Question 2

(a) $ex(n, H)$ is the largest integer m such that there exists an H-free graph on n vertices with m edges.

(b) Let G be a $K_{s,t}$-free graph on n vertices. W.l.o.g we may assume that G has minimum degree $s-1$ since adding edges incident to vertices of degree at most $s-2$ increases the number of edges of G but does not creates a $K_{s,t}$. Let $m = |E(G)|$ and let $\mathcal{K}$ be the set of $K_{s,1}$ subgraphs of G. Every vertex $v \in V(G)$ is incident to exactly $\binom{d(v)}{s}$ many $K_{s,1}$. Thus, $|\mathcal{K}| = \sum_{v \in V} \binom{d(v)}{s}$. In addition, since G is $K_{s,t}$-free, it must be the case that for every set of S of s vertices there exist at most $t-1$ vertices v such that the edges in $\{v\} \times S$ span a $K_{s,1}$. Thus $(t-1)\binom{n}{s} \geq |\mathcal{K}|$. Therefore,

$$(t-1)\binom{n}{s} \geq |\mathcal{K}| = \sum_{v \in V} \binom{d(v)}{s} \geq n\binom{\sum_{v \in V} d(v)/n}{s} = n\binom{2m/n}{s} \geq n\frac{(2m/n - s)^s}{s!}. \tag{1}$$

At the second inequality we used that the binomial coefficient $\binom{x}{s}$ is convex for $x \geq s-1$ and in addition that all the vertex degrees of G are greater than $s-1$. (1) implies,

$$\frac{(2m/n-s)^s}{s!} \leq (t-1)\binom{n}{s} \leq (t-1)\frac{n^s}{s!}.$$

Rearranging the above inequality gives,

$$m \leq \frac{1}{2}(t-1)^{1/s}n^{2-1/s} + \frac{sn}{2} \leq (t+s)n^{2-1/s}.$$

Therefore any $K_{s,t}$-free graph on n vertices can have at most $(t+s)n^{2-1/s}$ edges. Equivalently $ex(K_{t,s}) \leq (t+s)n^{2-1/s}$. □

Question 3

(a) For a graph G and two disjoint subsets $A, B \subseteq V(G)$ of the vertices let $e(A,B)$ denote the number of edges between A and B and let $d(A,B) := e(A,B)/|A||B|$. The pair $\{A,B\}$ is called ϵ-regular if for every $A' \subseteq A$ with $|A'| \geq \epsilon|A|$ and every $B' \subseteq B$ with $|B'| \geq \epsilon|B|$ we have $|d(A',B') - d(A,B)| \leq \epsilon$.

(b) For $X \in \{B,C\}$ let A_X be the set of vertices in A with less than $(d(A,X)-\epsilon)|X|$ many neighbors in X. Then,

$$|d(A,X) - d(A_X,X)| < d(A,X) - \frac{|X|(d(A,X)-\epsilon)|A_X|}{|X||A_X|} = \epsilon.$$

Since the pair $\{A,X\}$ is ϵ regular, due the definition of ϵ-regularity, it must be the case that $|A_X| \leq \epsilon|A|$. Let $A' = A \setminus (A_B \cup A_C)$. Then,

$$|A'| \geq |A| - |A_B| - |A_C| \geq (1-2\epsilon)|A|.$$

For $a \in A'$ let B_a, C_a be the set of neighbors of a in B and C respectively. Also let T_a be the set of triangles spanned by $A \times B \times C$ that are incident to a. Then, since every triangle in T_a corresponds to a unique edge in $B_a \times C_a$ we have $|T_a| = e(B_a, C_a)$. In addition for $X \in \{B,C\}$, since $a \notin A_X$, we have that

$$|X_a| \geq (d(A,X)-\epsilon)|X| \geq (2\epsilon - \epsilon)|X| = \epsilon|X|. \tag{2}$$

Thus, $B_a \subseteq B$ with $|B_a| \geq \epsilon|B|$, $C_a \subseteq C$ with $|C_a| \geq \epsilon|C|$ and the pair $\{B,C\}$ is ϵ-regular. Thus,

$$\epsilon \geq |d(B_a,C_a) - d(B,C)| = \left|\frac{e(B_a,C_a)}{|B_a||C_a|} - d(B,C)\right|$$

which implies that

$$e(B_a,C_a) \geq (d(B,C)-\epsilon)|B_a||C_a| \geq (d(B,C)-\epsilon)(d(A,B)-\epsilon)|B|(d(A,C)-\epsilon)|C|.$$

Therefore the number of triangles in G is at least

$$\begin{aligned}
&\sum_{a\in A'} |T_a| = \sum_{a\in A'} e(B_a, C_a)\\
&\geq \sum_{a\in A'} (d(A,B)-\epsilon)(d(A,C)-\epsilon)(d(B,C)-\epsilon)\cdot|B|\cdot|C|\\
&(1-2\epsilon)(d(A,B)-\epsilon)(d(A,C)-\epsilon)(d(B,C)-\epsilon)|A|\cdot|B|\cdot|C|.
\end{aligned}$$

□

Question 4 Let c be a blue/red coloring of $\mathbb{N}$ and let $n = W(2,31)$. Due the definition of $W(2,31)$ we know that there exist $a, d \in \mathbb{N}$ such that the set $S = \{a, a+d, a+2d, ..., a+30d\} \subseteq [n]$ is monochromatic. W.l.o.g we may assume that S is colored blue. Now consider the set $S_d = \{2d, 4d, 12d\}$. If there exists $s \in S_d$ that is colored blue then, since $a + 2.5s \in S$, $x = a, y = s, z = a + 2.5s$ gives a monochromatic solution to $x + 2.5y = z$. Otherwise all the elements of S_d are colored red in which case $x = 2d, y = 4d, z = 12d$ is a monochromatic solution of $x + 2.5y = z$. □

Question 5
For $p \in \{2,3\}$ let $\mathcal{F}_p = \{F \in \mathcal{F} : |F| \neq 0 \mod p\}$. Then, $F, F' \in \mathcal{F}_p$ implies that $|F|, |F'| \neq 0 \mod p$ and $|F \cap F'| = 0 \mod 6$, hence $|F \cap F'| = 0 \mod p$. Therefore $\mathcal{F}_p$ satisfies the conditions of the Mod-p-Town Theorem which gives that $|\mathcal{F}_p| \leq n$. Finally note that $F \in \mathcal{F}$ implies $|F| \neq 0 \mod 6$. Thus either $|F| \neq 0 \mod 2$, in which case $F \in \mathcal{F}_2$, or $|F| \neq 0 \mod 3$ and $F \in \mathcal{F}_3$. Therefore $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_3$ and

$$|\mathcal{F}| \leq |\mathcal{F}_2| + |\mathcal{F}_3| \leq 2n.$$

□

Question 6
(a) Let S be the set of points in $\mathbb{R}^n$ with 2 coordinates equal to 1 and $n-2$ coordinates equal to 0. Let $s_1, s_2 \in S$, $s_1 \neq s_2$. Then s_1, s_2 can have 0 or 1 common coordinates equal to 1. Therefore $|s_1 - s_2|_2 \in \{\sqrt{2}, \sqrt{4}\}$ and S is a two-distance set. (Note that by considering the set corresponding to S in $\mathbb{R}^{n+1}$ and observing that S lies in an n-dimensional hyperplane we can show that $m(n) \geq (n+1)n/2$.)

(b) Let S be a two-distance set in $\mathbb{R}^n$ and let $a, b > 0$ be the two distances that are achieved by the elements of S. For $s \in S$ define the polynomial $p_s : \mathbb{R}^n \mapsto \mathbb{R}$ given by

$$p_s(x) = (|x-s|_2^2 - a^2)(|x-s|_2^2 - b^2).$$

Then, $p_s(s) = (|s-s|_2^2 - a^2)(|s-s|_2^2 - b^2) = a^2b^2$. In addition $|s - s'|_2 \in \{a, b\}$ for $s' \in S, s \neq s'$ and therefore $p_s(s') = 0$. Therefore the set of polynomials $\mathcal{P} = \{p_s(x) : s \in S\}$ is linearly independent over $\mathbb{R}$.

$$p_s(x) = (|x-s|_2^2 - a^2)(|x-s|_2^2 - b^2) = \sum_{i\in n}[(x_i - s_i)^2 - a^2]\sum_{j\in n}[(x_j - s_j)^2 - b^2]$$

Thus $\mathcal{P}$ lies in the ring of polynomials over $\mathbb{R}$ that is spanned by $B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4$ where $B_0 = \{1\}$, $B_1 = \{x_i : i \in [n]\}$, $B_2 = \{x_ix_j : i, j \in [n]\}$,

$B_3 = \{x_i x_j x_h : i, j, h \in [n]\}$ and $B_4 = \{x_i x_j x_h x_l : i, j, h, l \in [n]\}$. Since $\mathcal{P}$ is linearly independent we have

$$|S| = |\mathcal{P}| \leq dim(span(B)) \leq \sum_{i=0}^{4} |B_i| \leq 5n^4.$$

(c) We note that for $s \in S$ $p_s(x)$ can be written as

$$\begin{aligned} p_s(x) &= \sum_{i \in n} [(x_i - s_i)^2 - a^2] \sum_{j \in n} [(x_j - s_j)^2 - b^2] \\ &= \sum_{i \in n} -2 s_i x_i \sum_{j \in n} x_j^2 + \sum_{i \in n} x_i^2 \sum_{j \in n} x_j^2 + q_s(x), \end{aligned}$$

where $q_s(x)$ is some quadratic polynomial that may depends on s. Let $B_5 = \{x_i \sum_{j \in n} x_j^2 : i \in [n]\}$ and $B_6 = \{(\sum_{j \in n} x_j^2)^2\}$. Then $\mathcal{P}$ lies in the ring of polynomials over $\mathbb{R}$ that is spanned by $B' = B_0 \cup B_1 \cup B_2 \cup B_5 \cup B_6$. Since $\mathcal{P}$ is linearly independent we have

$$|S| = |\mathcal{P}| \leq dim(span(B')) \leq \sum_{i=0}^{2} |B_i| + |B_5| + |B_6| \leq 5n^2.$$