Extremal Combinatorics Wi 2017–18
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Asymptotics

In the exercise class for Discrete Math I review we struggled a bit with [1] Exercise 4 and the asymptotic comparisons. So, I will sketch some of the proofs here.

(1) $n! = o(n^n)$.
We can use the estimate $n! \leq en\left(\frac{n}{e}\right)^n$, which is true for all positive integers n. Then $\frac{n!}{n^n} \leq en\left(\frac{1}{e}\right)^n$, and the limit of the right hand side is 0. Hence $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.
Alternatively, we have

$$\frac{n!}{n^n} = \Big(\prod_{i=1}^{n/2} \frac{i}{n}\Big)\Big(\prod_{i=n/2+1}^{n} \frac{i}{n}\Big) \leq \prod_{i=1}^{n/2} \frac{i}{n} \leq \frac{1}{2^{n/2}}.$$

Or even more simply, we can use $n! \leq n^{n-1}$.

(2) $n^n = o(2^{n^2})$.
Let $f = n^n$ and $g = 2^{n^2}$. Then $\log f = n \log n$ and $\log g = n^2$. We have $\log(f/g) = \log f - \log g = -\log g(1 - \log f/\log g) = -n^2(1 - \log n/n)$. Since $\lim_{n\to\infty} \log n/n = 0$, we get $\lim_{n\to\infty} \log(f/g) = -\infty$. This implies that $\lim_{n\to\infty} f/g = 0$.
Alternatively, we can use the identity $x = 2^{\log x}$ to write $f = 2^{n \log n}$. We then get

$$\frac{f}{g} = \frac{1}{2^{n^2 - n\log n}},$$

which approaches 0 as n approaches ∞.

(3) $2^{n^2} = o(2^{2^{(\log n)^2}})$.
Let $f = 2^{n^2}$ and $g = 2^{2^{(\log n)^2}}$. Again by using $x = 2^{\log x}$, we have $g = 2^{n^{\log n}}$ and hence

$$\frac{f}{g} = \frac{1}{2^{n^{\log n} - n^2}}.$$

Therefore, $\lim_{n\to\infty} f/g = 0$.

Note that if $f = o(g)$ then $f = O(g)$. Also, if $f = o(g)$ and $g = o(h)$ then $f = o(h)$. Another observation that simplifies things is that if $\lim_{n\to\infty}(\log g - \log f) = \infty$, then $f = o(g)$, which is what we have used in (2) and (3).

[1] or at least I did