# Exercise sheet 1

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012
Due date: May 2nd (Wednesday) by 8:30, at the end of the exercises.

**Definition:** A *projective plane* $\Pi$ is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is a set of elements called *points*, $\mathcal{L} \subseteq \in^{\mathcal{P}}$ is a family of subsets of points called *lines*, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an *incidence relation* between points and lines such that the following holds:

(i) $\forall \ell_1, \ell_2 \in \mathcal{L}, \ell_1 \neq \ell_2,\ \exists! p \in \mathcal{P}\ : (p, \ell_1), (p, \ell_2) \in \mathcal{I}$
(every pair of distinct lines is incident to a unique point),

(ii) $\forall p_1, p_2 \in \mathcal{P}, p_1 \neq p_2,\ \exists! \ell \in \mathcal{L}\ : (p_1, \ell), (p_2, \ell) \in \mathcal{I}$
(every pair of distinct points is incident to a unique line),

(iii) there are four points such that no three of them are incident to a single line (non-degeneracy)

A projective plane is called finite, if $\mathcal{P}$ is a finite set.

**Problem 1.** Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite projective plane. Prove that there is an integer $m \geq 2$ such that every point (line, respectively) of $\Pi$ is incident to exactly $m+1$ lines (points, respectively) and that $|\mathcal{P}| = m^2 + m + 1 = |\mathcal{L}|$. (The integer $m$ is called the *order* of the finite projective plane $\Pi$.)

**Remark:** The prime power conjecture for projective planes is a very famous open problem saying that a projective plane of order $m$ exists if and only if $m$ is a prime power. It is known that there is no projective plane of order 6 and 10, where the latter has only recently be proved. However, the status of projective plane of order 12 remains open. The easy direction of the conjecture is verified in the next exercise.

**Definition:** (Projective plane $PG(q, 2)$ over the finite $q$-element field $\mathbb{F}_q$)

- Points: Let $\sim$ be the equivalence relation defined on $\mathbb{F}_q^3 \setminus \{(0,0,0)\}$, where two triples are in relation if they are nonzero constant multiples of each other: $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if there is a $c \in \mathbb{F}_q^*$ such that $y_i = cx_i$ for $i = 0, 1, 2$. The equivalence class of $(x_0, x_1, x_2)$ is denoted by

$$[x_0, x_1, x_2] = \left\{ (cx_0, cx_1, cx_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\ :\ c \in \mathbb{F}_q^* \right\}.$$

The pointset $\mathcal{P}$ of $PG(q, 2)$ consists all these equivalence classes.

$$\mathcal{P} = \{[x_0, x_1, x_2] : (x_0, x_1, x_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\}.$$

- Lines: Given a triple $(a_0, a_1, a_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$ we define the line $L(a_0, a_1, a_2)$ as follows:

$$L(a_0, a_1, a_2) := \left\{ [x_0, x_1, x_2] \in \mathcal{P} \ : \ a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \right\} .$$

  The family $\mathcal{L}$ of lines in $PG(q, 2)$ consists of all these lines $L(a_0, a_1, a_2)$.

- Incidence: The incidence relation $\mathcal{I}$ is defined by the containment: $(p, \ell) \in \mathcal{I}$ if $p \in \ell$.

**Problem 2.** Verify that $PG(q, 2)$ satisfies the axioms of a projective plane and show that its order is $q$.

**Definition:** (Projective $d$-space $PG(q,d)$ over $\mathbb{F}_q$) The *points* of $PG(q,d)$ are the equivalence classes of the equivalence relation $\sim$ defined on $\mathbb{F}_q^{d+1} \setminus \{(0, \ldots, 0)\}$ the following way: $(x_0, x_1, \ldots, x_d) \sim (y_0, y_1, \ldots, y_d)$ if $\exists c \in \mathbb{F}_q^*$ such that $y_i = c x_i$ for all $i = 0, 1 \ldots, d$. We denote the equivalence class of $(x_0, x_1, \ldots x_d)$ by

$$[x_0, \ldots, x_d] = \left\{ (cx_0, \ldots, cx_d) \in \mathbb{F}_q^{d+1} \setminus \{(0, \ldots, 0)\} \ : \ c \in \mathbb{F}_q^* \right\} .$$

A *k-dimensional subspace* in $PG(q,d)$ is the set of all points $[x_0, x_1, \ldots, x_d]$ whose coordinates satisfy $d-k$ linearly independent homogeneous linear equations

$$\begin{array}{ccccccc} a_{10}x_0 & + & \cdots & + & a_{1d}x_d & = & 0 \\ \vdots & & \ddots & & \vdots & = & \vdots \\ a_{d-k,0}x_0 & + & \cdots & + & a_{d-k,d}x_d & = & 0 \, , \end{array}$$

with coefficients $a_{ij} \in \mathbb{F}_q$.

A $(d-1)$-dimensional subspace of $PG(q,d)$, that is the set of all points satisfying a single linear equation $a_0 x_0 + \cdots + a_d x_d = 0.$, is called a *hyperplane*. A $k$-dimensional subspace is therefore the intersection of $d-k$ hyperplanes. The 1-dimensional subspaces are called *lines* and it is easy to check that the 0-dimensional subspaces are the points of $PG(q,d)$.

**Problem 3.** Compute the number of points in a $k$-dimensional subspace in $PG(q,d)$.

**Problem 4.** Define $ex(n, m, H)$ as the largest number $e$, such that there is an $H$-free bipartite graph with partite sets of size $n$ and $m$ respectively containing $e$ edges. Show that

$$ex(q^2 + q + 1, q^2 + q + 1, K_{2,2}) = (q^2 + q + 1)(q + 1)$$

for every prime power $q$, i.e. the graph from Construction 1 is *an* optimal construction.