Exercise Sheet 2.

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012

Due date: May 9th (Wednesday) by 8:30, at the beginning of the exercises.

Problem 1. Consider the graph G from Construction 2 whose vertex set is $\mathbb{F}_p^2 \setminus \{(0,0)\}$ and where two vectors (a_1, a_2) and (b_1, b_2) are adjacent if

$$a_1 b_1 + a_2 b_2 = 1.$$

Calculate the number of loops in G *exactly* and conclude that the number of its edges is $n^{3/2}/2 - O(\sqrt{n})$.

(*Hint:* Conjecture the answer for the number of loops by experimenting on small primes. For the solution, consider the more general problem of determining $N(\alpha) := |\{(x,y) \in \mathbb{F}_q^2 : x^2 + y^2 = \alpha\}|$. Start by determining $N(0)$, then show that N is constant on $QR(q)$ and on $QNR(q)$. Find $N(1)$ by double-counting the sum $\sum_{w \in QR(q)} N(w)$.)

Problem 2. Define the *polarity graph* G (Construction 3) on the points of the projective plane $PG(q,2)$ in the following way. Let $V(G) = \{[x_0, x_1, x_2] \in \mathcal{P}\}$ and $E(G) = \{\{x,y\} : x \neq y, x_0 y_0 + x_1 y_1 + x_2 y_2 = 0\}$.

Show that the polarity graph contains exactly $q+1$ vertices of degree q.

Problem 3. Prove that for any $a \in \mathbb{F}_p^3$ the sphere $S_\alpha(a)$ contains either $p^2 - p$ or $p^2 + p$ points depending on whether α and -1 are quadratic residues or not. Conclude the exact number of edges in the Brown graph (the unit-distance graph in $\mathbb{F}_p^3$).

Can you give a general exact formula for the number of solutions to $x_1^2 + \cdots + x_k^2 = \beta$, for any fixed $k \in \mathbb{N}, \beta \in \mathbb{F}_p$?

Problem 4. Prove that the chromatic number of the unit distance graph of the eucledian plane $\mathbb{R}^2$ lies between 4 and 7. (For the lower bound try to find a 4-chromatic subgraph on seven vertices. For the upper bound try to find a proper coloring based on an appropriate regular lattice.)

Remark. There are no better bounds known on the chromatic number. Any improvement of even just 1 over the above bounds would represent a major progress and carries a prize of \$1000.