Exercise sheet 3

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012
Due date: May 16th (Wednesday) by 8:30, at the beginning of the exercise class.

Problem 1. Let k be an arbitrary field. Prove that if $-\alpha \in k$ *is* a square, then the corresponding sphere-graph (defined in the 3-dimensional space over k) not only contains a $K_{3,3}$, but also a $K_{n^{1/3},n^{1/3}}$. (In case k is an infinite field we mean a $K_{|k|,|k|}$.)

In particular, our heuristics for Brown's $K_{3,3}$-free construction would fail badly in the complex 3-space.

Question. Can you also find a $K_{n^{1/3},n^{2/3}}$?

Problem 2. Recall the following theorem from the lecture.
Theorem *Let* $c_0, c_1, \ldots, c_n \in \mathbb{F}_p^*$. *Then we have*

$$|N(c_0, \ldots, c_n) - p^{n-1}| \leq (d_1 \cdots d_n) p^{(n-1)/2} \left(\frac{p}{p-1}\right)^{n/2}.$$

Prove that $N(c_0, c_1, \ldots, c_n)$ cannot deviate by "much" from the average even if $c_0 = 0$.

Problem 3. Prove that in Brown's graph roughly half of the triples of vertices have two common neighbors and the other half has none. Even more: describe explicitly those triples of vertices which do not have a common neighbor!

Problem 4. A natural thought to extend the idea of the Brown graph to $K_{4,4}$- or $K_{4,1000}$-avoiding dense graphs is the following. Instead of three dimensions let us take four, i.e. our vertex set is $\mathbb{F}_p^4$. Let the neighborhood of a vertex x be determined by a four-dimensional sphere around it, in particular y is adjacent to x if $\sum_{i=1}^4 (y_i - x_i)^2 = 1$. According to the above Theorem, our graph has roughly $cn^{7/4}$ edges — the conjectured truth. Prove, however, that this graph contains a $K_{p,p}$.

Also show that even taking a higher degree surface of the form $\sum_{i=1}^4 (y_i - x_i)^{1000} = 1$ as the neighborhood of x instead of the sphere would not help us. (Note that the above theorem ensures that this graph as well has roughly the correct number $cn^{7/4}$ of edges.)