Exercise Sheet 4.

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012

Due date: May 23rd (Wednesday) by 8:30, at the beginning of the exercise class.

Problem 1. Generalize the upper bound proof of Füredi (for $K_{3,3}$-free graphs) and show that $ex(n, K_{4,4}) \lessapprox \frac{1}{2}n^{7/4}$. (Hint: Instead of lower bounding $\sum \binom{x_i}{3}$ in terms of $(\sum x_i)^3$ (which follows from the convexity of $\binom{x}{3}$) you might want to bound it from below in terms of the product of $\sum \binom{x_i}{2}$ and $\sum x_i$.)

Problem 2. Let the vertex set of a graph G be $\mathbb{F}_p^4$. Let (a, b, c, d) be adjacent to (a', b', c', d') if and only if $(a + a')(b + b')(c + c')(d + d') = 1$. Prove that G contains a $K_{n^{1/4},n^{1/4}}$.

Problem 3. The *k-color Ramsey number* $R_k(G)$ is the smallest integer m, such that no matter how the edges of K_m are colored with k colors, there exists a monochromatic copy of G.

Show that $R_k(K_{3,3}) = (1 + o(1))k^3$.

(*Hint:* For the lower bound use the projective norm-graphs and the Key Lemma.)

Problem 4. Use the projective norm-graphs together with Füredi's idea (which improves the $K_{2,2}$-free Construction 2 to a $K_{2,s}$-free construction) to give a $K_{3,s}$-free construction whose number of edges comes within a factor $\sqrt[3]{2} + o(1)$ of the KST upper bound of $ex(n, K_{3,s})$ for every $s \geq 3$ of the form $s = 2r^2 + 1$, $r \in \mathbb{Z}$. (The $o(1)$ above is understood as $s \to \infty$.)

More generally, prove that for any fixed t

$$\lim_{s\to\infty} (\liminf_{n\to\infty} ex(n, K_{t,s})n^{-(2-1/t)}) = \infty.$$

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