Exercise Sheet 5.

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012
Due date: June 13th (Wednesday) by 8:30, at the beginning of the exercise class.

Problem 1. Derive the Moore bound for even girth. That is, show that if G is a d-regular graph with girth $2k$, then

$$n(G) \geq 2\sum_{i=0}^{k-1}(d-1)^i.$$

Problem 2. Let $\mathcal{P}$ be the set of points $x = (x_{-2}, x_{-1}, x_0, x_1, x_2)$ of the projective four-space $PG(q, 4)$ and let Q_4 be the quadratic surface defined for Benson's C_6-free graph: $Q_4 = \{x \in \mathcal{P} \ : \ x_0^2 + x_1x_{-1} + x_2x_{-2} = 0\}$.

Prove that the following properties hold:

(*i*) $|Q_4| = q^3 + q^2 + q + 1$,

(*ii*) Q_4 contains $q^3 + q^2 + q + 1$ lines,

(*iii*) every line of Q_4 contains $q + 1$ points,

(*iv*) every point is contained in $q + 1$ lines of Q_4.

Problem 3. Show that no 3-dimensional subspace of $PG(q, 6)$ is fully contained in the quadratic surface

$$Q_6 = \{x \in PG(q,6) \ : \ x_0^2 + x_1x_{-1} + x_2x_{-2} + x_{-3}x_3 = 0\},$$

which we defined for Benson's C_{10}-free construction.

Let $\mathcal{P}$ be a set of *points*, and let $\mathcal{L}$ be a set of *lines*. A subset $I \subseteq \mathcal{P} \times \mathcal{L}$ is called an *incidence relation* on $(\mathcal{P}, \mathcal{L})$, and then we say that the triple $(\mathcal{P}, \mathcal{L}, I)$ is a *(rank two) geometry.*

The *incidence graph* Γ of the geometry is defined on $V(\Gamma) = \mathcal{P} \cup \mathcal{L}$, and its edge set is $E(\Gamma) = \{pl : p \in \mathcal{P}, l \in \mathcal{L}, (p, l) \in I\}$.

We say that the function $\pi : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ is a *polarity* of geometry $(\mathcal{P}, \mathcal{L}, I)$ if

(i) $\mathcal{P}^\pi = \mathcal{L}$ and $\mathcal{L}^\pi = \mathcal{P}$,

(ii) for every point p and every line l, we have that p and l are incident if and only if their polar images are incident, i.e $(p, l) \in I \iff (l^\pi, p^\pi) \in I$,

(iii) $\pi^2 = \mathrm{id}$.

In case a polarity exists we can define the *polarity graph* Γ^π of the geometry as follows: $V(\Gamma^\pi) = \mathcal{P}$, and $E(\Gamma^\pi) = \{p_1p_2 : p_1 \neq p_2, (p_1, p_2^\pi) \in I\}$.

We say that $p \in \mathcal{P}$ is an *absolute point* of π if $(p, p^\pi) \in I$. The set of all absolute points is denoted by $N_\pi = \{p : \ p \text{ is absolute point of } \pi\}$.

Problem 4. Prove the following properties.

(i) If $p \in N_\pi$, then we have $d_{\Gamma^\pi}(p) = d_\Gamma(p) - 1$. Otherwise, $d_{\Gamma^\pi}(p) = d_\Gamma(p)$.

(ii) $|V(\Gamma^\pi)| = \frac{1}{2}|V(\Gamma)|$ and $|E(\Gamma^\pi)| = \frac{1}{2}\left(|E(\Gamma)| - |N_\pi|\right)$,

(iii) If $C_{2k+1} \subseteq \Gamma^\pi$, then $C_{4k+2} \subseteq \Gamma$,

(iv) If $C_{2k} \subseteq \Gamma^\pi$, then there are two copies of C_{2k} in Γ such that one is the polar dual of the other,

(v) $g(\Gamma^\pi) \geq \frac{1}{2}g(\Gamma)$.