Exercise Sheet 6.

Tibor Szabó
Discrete Mathematics III — Constructive Combinatorics, Summer 2012
Due date: June 20th (Wednesday) by 8:30, at the beginning of the exercise class.

Problem 1. Let $\mathcal{P}_* = \{(a,b,c) : a,b,c \in \mathbb{F}_q\}$, $\mathcal{L}_* = \{[d,e,f] : d,e,f \in \mathbb{F}_q\}$, and $((a,b,c),[d,e,f]) \in I_*$ if and only if $e - b = da$, $f - c = ea$. Prove that the incidence graph of the above geometry $(\mathcal{P}_*, \mathcal{L}_*, I_*)$ is nothing else but Wenger's C_6-free construction.

Problem 2. Let $q = 2^{2\alpha+1}$ and define π_* by

$$\begin{aligned} \pi_* : (a,b,c) &\mapsto \left[a^{2^{\alpha+1}}, (ab)^{2^\alpha} + c^{2^\alpha}, b^{2^{\alpha+1}}\right], \\ \pi_* : [d,e,f] &\mapsto \left[d^{2^\alpha}, f^{2^\alpha}, (df)^{2^\alpha} + e^{2^{\alpha+1}}\right]. \end{aligned}$$

- Prove that π_* is a polarity.
- Prove that set of absolute points is

$$N_{\pi_*} := \{(a, b, a^{2^{\alpha+1}+2} + ab + b^{2^{\alpha+1}}) : a,b \in \mathbb{F}_q\}.$$

- Show that Γ_*^π is C_6-free and conclude that $ex(n, C_6) \geq \frac{1}{2}n^{4/3} - \frac{1}{2}n^{2/3}$.
- Show that Γ_*^π is also C_3-free and C_4-free. (We needed these properties in the lecture to produce an even denser C_6-free graph.)

Problem 3. Finish the proof of the theorem of Füredi, Naor, and Verstraete from the lecture by calculating the expected number of edges of H if A is selected randomly: independently put each vertex of G into A with the same probability p. (This eventually should prove that $ex(n, C_6) \gtrsim 0.534n^{4/3}$.)