Exercise Sheet 8.

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Discrete Mathematics III — Constructive Combinatorics, Summer 2012
Due date: July 10th (Tuesday) by 12:30, at the beginning of the exercise class.

In the following four exercises let H be an abelian group .

1. Let $\chi \in \hat{H}$ be an arbitrary character and $a \in H$ be an arbitrary group element.

(i) $\chi(a)$ is a $|H|^{th}$ root of unity.

(ii) $\chi(-a) = \chi(a)^{-1} = \overline{\chi(a)}$.

2. Prove that $(\hat{H}, \cdot)$ is an abelian group where the operation is defined as follows.

$$(\chi \cdot \psi)(a) := \chi(a)\psi(a).$$

3. Prove that if $H = H_1 \times H_2$, then $\hat{H} \cong \hat{H}_1 \times \hat{H}_2$.

4. Prove that for any $\chi, \psi \in \hat{H}$,

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{otherwise.} \end{cases}$$

5. Show that if a sample space $S \subseteq \{0,1\}^N$ is ϵ-close to independent then it is also $\epsilon 2^N$-unbiased with respect to linear tests. Construct a sample space that shows the statement being best possible (for all sensible values of the parameters N and ϵ).

6. Prove that for *every* k the Frankl-Wilson graphs provide a k-Ramsey graph on $k^{\Omega\left(\frac{\log k}{\log\log k}\right)}$ vertices.

7. The proof of the following theorem is an immediate generalization of the claim we (will) have in the lecture about the clique number of the Frankl-Wilson graph. (Think this over!)

Theorem Let L be a set of integers with $|L| = s$. Let $B_1, \ldots, B_t \in 2^{[n]}$ be a uniform L-intersecting family, i.e. all $|B_i|$ have the same size and $|B_i \cap B_j| \in L$ for every $i \neq j$. Then $t \leq \sum_{i=0}^{s} \binom{n}{i}$. $\square$

Generalize this statement further to *arbitrary* L-intersecting families, i.e. derive the same conclusion when the $|B_i|$ are not necessarily all equal. (Hint: Select the functions $\tilde{f}_i$ more carefully and use Lemma 5.6.1 in its full power.)