By Jensen's inequality we infer

$$2n\binom{\bar{d}(G)}{2} \leq n\bar{d}(G)(n-1)^{2/3} + 2(n-1)n,$$

implying

$$\bar{d}(G) - 1 \leq (n-1)^{2/3} + \frac{2(n-1)}{\bar{d}(G)}.$$

If $\bar{d}(G) \leq (n-1)^{2/3}$, then we are done. Otherwise by the above we have

$$\bar{d}(G) \leq (n-1)^{2/3} + \frac{2(n-1)}{\bar{d}(G)} + 1 \leq (n-1)^{2/3} + 2(n-1)^{1/3} + 1.$$

Hence, with $e(G) = n\bar{d}(G)/2$ we concluded the proof of

$$ex(n, K_{3,3}) \leq \frac{1}{2}n^{5/3} + n^{4/3} + \frac{1}{2}n.$$

□

Exercise 2.11 *Improve the KST-upper-bound a bit. Show that for arbirary $s \geq 3$, we have $ex(n, K_{3,s}) \lessapprox \frac{\sqrt[3]{s-2}}{2}n^{5/3}$.*

Exercise 2.12 *Generalize the proof above and show that $ex(n, K_{4,4}) \lessapprox \frac{1}{2}n^{7/4}$. (Hint: Instead of lower bounding $\sum \binom{x_i}{3}$ in terms of $(\sum x_i)^3$ (which follows from the convexity of $\binom{x}{3}$) you might want to bound it from below in terms of the product of $\sum \binom{x_i}{2}$ and $\sum x_i$.)*

Open Problem. The asymptotics of $K_{3,s}$ is not known for any $s > 3$. There are infinitely many values of s for which the upper and lower bounds are within a constant factor of $\sqrt[3]{2}$ of each other (we will discuss these results later), but there are also infinitely many values s where this constant factor separation is $\sqrt[3]{s-2}$.
Any improvement would be very interesting. The value of $ex(n, K_{3,4})$ is the first unknown.

2.3 Forbidding $K_{t,f(t)}$

So far in all we've seen on the front of dense $K_{t,s}$-free constructions, the smaller of the parameters t and s was at most 3. So what does become so hard when t and s are both at least 4? Phrasing it mysteriously, besides us being not creative enough, the problem is that $4 = 2 + 2$.

Typical $K_{t,s}$-free constructions live in the t-dimensional space (over a finite field). The vertex set is usually chosen to be the space itself, and the neighborhood of each vertex is defined by a surface. On the one hand, one has to prove that the surfaces contain the appropriate number, $\approx p^{t-1}$, of points; this is sometimes easier, sometimes harder, but it can always be done. On the other hand, one must also show that the intersection of any t of the selected p^t surfaces contains at most $s-1$ points. The exercises leading up to this section tried to demonstrate that this is the more problematic issue, and in fact the most critical point is to show that these t-wise intersections contain only *finitely many* points. Then it is usually only a bonus that one is able to bound this finite number (by $t-1$, or $s-1$, whatever s was in the particular problem). In the exercises we investigated promising constructions, which broke down in a strong way: we found complete bipartite graphs whose order tended to infinity with the order of the graph.

When one takes t "average" surfaces in t-space one expects that their intersection is 0-dimensional (finite). However, experience shows that for $t \geq 4$ it is hard to select p^t surfaces in the t-dimensional space such that any t of them has a 0-dimensional (finite) intersection.

Part of the problem are degeneracies: a line (or some low degree curve) is part of the surface. The next exercise demonstrates some more what can occur when we step into the fourth dimension. It is a prelude for what is coming in the following section.

Exercise 2.13 *Let the vertex set of a graph G be $\mathbb{F}_p^4$. Let (a,b,c,d) be adjacent to (a',b',c',d') if and only if $(a+a')(b+b')(c+c')(d+d')=1$. Prove that G contains a $K_{n^{1/4},n^{1/4}}$.*

The problem here arises because the variables could be "separated" from each other; there are four of them and $4=2+2$, each pair is responsible for a curve, and these create the large complete bipartite graph. This difficulty is overcome in the next section by introducing much-much higher degree in the equation, but still keeping the simple structure of the equation in the exercise. The key to this is the existence of the *Frobenius automorphism.* If the characteristic of a field $\mathbb{F}$ is q, then the mapping $X \to X^q$, $X \in \mathbb{F}$, is an automorphism of $\mathbb{F}$, the Frobenius automorphism. Indeed, the function is of course a bijection since q and $|\mathbb{F}^*|$ are relatively prime. The multiplication is clearly interchangeable with the mapping, while the addition is interchangeable since for any $Z \in \mathbb{F}$, $qZ=0$, and hence

$$(X+Y)^q = X^q + \binom{q}{1}X^{q-1}Y + \cdots + \binom{q}{q-1}XY^{q-1} + Y^q = X^q + Y^q.$$

In the following we will define a sequence of dense $K_{t,s}$-free graphs for *arbitrary* t. Although we are not able to say anything new about $ex(n, K_{t,s})$ when $s=t$, we will at least determine its order of magnitude when s is *some* function of t. Namely, the *norm-graphs*, defined in the next section, are $K_{t,t!+1}$-free and have $cn^{2-1/t}$ edges thus matching the Kővári-Sós-Turán upper bound. Later $t!+1$ will be improved to $(t-1)!+1$ by a modified construction called the *projective norm-graph.*

2.3.1 The norm-graphs

Let q be a prime power and let t be a positive integer. The *norm-graph* $G_{q,t} = G$ is defined as follows. Let $V(G) = \mathbb{F}_{q^t}$, $E(G) = \{\{A,B\} : N(A+B) = 1\}$, where $N(\cdot) : \mathbb{F}_{q^t} \to \mathbb{F}_q$ is the norm function[1]

$$N(A) = A \cdot A^q \cdot \dots \cdot A^{q^{t-1}} = A^{(q^t-1)/(q-1)}.$$

We have $|V(G)| = q^t =: n$. For a fixed $A \in V(G)$, the number of solutions X to the equation $N(X+A) = 1$ is exactly $\frac{q^t-1}{q-1}$. (Remember, for any $\alpha \in \mathbb{F}$ in a finite field $\mathbb{F}$, $|\{x : x^l = \alpha\}|$ is either 0 or $|\mathbb{F}^*|/(|\mathbb{F}^*|, l)$.) Hence, excluding the possible loops at those A where $N(2A) = 1$,

$$|E(G)| \geq \frac{1}{2} q^t \left(\frac{q^t-1}{q-1} - 1 \right) \geq \frac{1}{2} n^{2-1/t}.$$

In the remaining of the section we study why G is $K_{t,t!+1}$-free. What does the presence of a $K_{t,s}$ in G mean? If a $K_{t,s}$ is present in G, then there exist $D_1, \dots, D_t \in \mathbb{F}_{q^t}$ such that the system of equations

$$\begin{array}{rcl} (X+D_1)(X^q+D_1^q)\cdots(X^{q^{t-1}}+D_1^{q^{t-1}}) & = & 1 \\ (X+D_2)(X^q+D_2^q)\cdots(X^{q^{t-1}}+D_2^{q^{t-1}}) & = & 1 \\ \vdots \qquad\qquad \vdots & & \vdots \\ \vdots \qquad\qquad \vdots & & \vdots \\ (X+D_t)(X^q+D_t^q)\cdots(X^{q^{t-1}}+D_t^{q^{t-1}}) & = & 1 \end{array} \tag{2.5}$$

has s solutions $X \in \mathbb{F}_{q^t}$. Note that here we used the comfortable fact shown above, that the mapping $A \mapsto A^q$ is a field automorphism (the *Frobenius automorphism*), and thus, in particular, $(A+B)^q = A^q + B^q$.

To obtain a bound on the number of solutions we consider a much more general setup.

Lemma 2.8.1 (Key Lemma) *Let $\mathbb{F}$ be a field, and $a_{ij}, b_i \in \mathbb{F}$ for $1 \leq i,j \leq t$, such that $a_{i_1 j} \neq a_{i_2 j}$ if $i_1 \neq i_2$. Then*

$$\begin{array}{rcl} (x_1-a_{11})(x_2-a_{12}) \; \cdots \; (x_t-a_{1t}) & = & b_1 \\ (x_1-a_{21})(x_2-a_{22}) \; \cdots \; (x_t-a_{2t}) & = & b_2 \\ \vdots \qquad\qquad \vdots & & \vdots \\ \vdots \qquad\qquad \vdots & & \vdots \\ (x_1-a_{t1})(x_2-a_{t2}) \; \cdots \; (x_t-a_{tt}) & = & b_t \end{array} \tag{2.6}$$

has at most $t!$ solutions in $\mathbb{F}^t$.

[1] The *norm* of the field extension $\mathbb{F}_{q^l}$ over $\mathbb{F}_q$ is the map N_l defined on $\mathbb{F}_{q^l}$ by $N_l(A) = A \cdot A^q \cdots A^{q^{l-1}}$. We drop the subscript l throughout, as it will be apparent from the context. Clearly N is a multiplicative function: if $A, B \in \mathbb{F}_{q^l}$ then $N(AB) = N(A)N(B)$. From $N(A)^q = N(A)$ we infer that $N(A) \in \mathbb{F}_q$ for every $A \in \mathbb{F}_{q^l}$. Indeed, the roots of the polynomial $x^q - x$ are precisely the elements of $\mathbb{F}_q$, and it vanishes at $N(A)$.

Remark: 1. The Key Lemma is easily proved when $b_1 = \cdots = b_t = 0$. To satisfy the first equation one must select a factor $(x_{\pi(1)} - a_{1\pi(1)})$ to be 0. This can be done t ways. Once the variable $x_{\pi(1)} = a_{1\pi(1)}$ is fixed, one can simplify the remaining $t-1$ equations with the factor $(x_{\pi(1)} - a_{i\pi(1)})$, since $a_{1\pi(1)} \neq a_{i\pi(1)}$ by the condition of the lemma. We end up with $t-1$ equations of the same kind as in the lemma and can proceed by induction.
2. The statement of the Key Lemma is best possible, which is witnessed by the case when all $b_i = 0$. Indeed, $(a_{1\pi(1)}, \ldots, a_{t\pi(t)})$ is a (distinct) solution for every $\pi \in S_t$.
3. Observe that the scenario is similar to the one we faced during the solution of Exercise 2.13. There the four equations similar to (2.6) occasionally had $\sim |\mathbb{F}|$ solutions. Nevertheless, according to the Key Lemma, this could only occur if some coordinate of at least two of the four points agreed. In the setup of the norm-graph the Frobenius authomorphism helps us to avoid this degeneracy.

Corollary 2.9 *For any prime power q and integer $t \geq 1$ the norm-graph $G_{q,t}$ does not contain a $K_{t,t!+1}$. In particular*

$$ex(n, K_{t,t!+1}) = \Theta\left(n^{2-1/t}\right).$$

Proof. Applying the Key Lemma with $\mathbb{F} = \mathbb{F}_{q^t}$, $a_{ij} = -D_i^{q^{j-1}}$, $x_i = X^{q^{i-1}}$, and $b_i = 1$ we have the $t!$ bound on the number of solutions of the system (2.5). To check the condition of the Key Lemma we recall that the Frobenius automorphism is in particular a bijection, so for a fixed j the $-D_i^{q^{j-1}}$ are all distinct. □

To get the bound on the number of solutions of system (2.6) of the Key Lemma we make use of the following standard tool.

Claim 1 *Let K be an algebraically closed field, $A = K[x_1, \ldots, x_t]$, $f_1, \ldots, f_r \in A$, $B = K[f_1, \ldots, f_r]$, and*

$$F : K^t \to K^r, \quad F(x) := (f_1(x), \ldots, f_r(x)).$$

If A is integral over B and B is integrally closed in $QF(B)$, then for any $b \in K^r$

$$|F^{-1}(b)| \leq d := \dim_{QF(B)} QF(A).$$

Proof. Let $F^{-1}(b) = \{P_1, \ldots, P_s\} \subseteq K^t$ and choose a polynomial $g \in A$ such that $g(P_i)$ are all distinct (this is easy since K is infinite). Let $h(y) \in QF(B)[y]$ be the monic minimal polynomial of g over $QF(B)$. Then $q := deg(h) \leq d$, otherwise $\{1, g, g^2, \ldots g^d\} \subseteq QF(A)$ would be a linearly independent set over $QF(B)$, a contradiction.

The coefficients of h are integral over B, so they are from B (since B is integrally closed in $QF(B)$). Why are the coefficients of h integral over B? Since one of its roots, g is integral, integrality carries over to all its roots and the coefficients are polynomials of the roots of h. Indeed, if L is the splitting field of h, then any zero $g' \in L$ of h can be mapped to g by an automorphism of L which fixes $QF(B)$, so the integrality of g over B implies the integrality of g' over B.

So there exists coefficients $c_1, \dots c_q \in B$ such that

$$g^q + c_1 g^{q-1} + \dots c_q = 0.$$

If we substitute $P_1, \dots P_s$ into this polynomial equation, we obtain s equations over K.

Note that $c_j(P_i)$ does **not** depend on i, since $F(P_i)$ is constant and B is generated by the coordinate functions of F. Let $e_j = c_j(P_i)$.

Since $g(P_1), \dots, g(P_s)$ are distinct solutions to the equation

$$g^q + e_1 g^{q-1} + \dots e_q = 0,$$

we have $s \le q \le d$. □

Proof. (of the Key Lemma) We naturally set $f_i(x) = (x_1 - a_{i1})(x_2 - a_{i2}) \cdots (x_t - a_{it})$ and $r = t$. By Claim 1 we need to prove

- $A = K[x_1, \dots, x_t]$ is integral over $B = K[f_1, \dots, f_r]$ (this implies that A is a finitely generated B-module, since it is a finitely generated B-algebra),
- B is integrally closed in $QF(B)$, and
- $\dim_{QF(B)} QF(A) = t!$.

Integrality of A over B: We use induction on t.
The case $t = 1$ is trivial: $A = K[x_1] = K[x_1 - a_{11}] = B$.

Let $t > 1$, and let $R \ge B$ be an arbitrary valuation ring. Since the integral closure of B is the intersection of all valuation rings containing it, we are done once we prove $R \ge A$.

Assume $x_t \notin R$. This implies $x_t - a_{it} \notin R$ for all $i = 1, \dots, t$. Then

$$\frac{1}{x_t - a_{it}} \in \mathfrak{m} \lhd R \quad \overset{f_i \in R}{\Longrightarrow} \quad g_i := \frac{f_i}{x_t - a_{it}} \in \mathfrak{m},$$

for all $i = 1, \dots, t$, where $\mathfrak{m}$ is the unique maximal ideal of the valuation ring R.

By induction $K[x_1, \dots, x_{t-1}]$ is integral over $K[g_1, \dots, g_{t-1}]$, we have

$$g_1, \dots, g_{t-1} \in R \quad \Rightarrow \quad K[x_1, \dots, x_{t-1}] \le R.$$

The polynomials $g_1, \dots, g_{t-1}, g_t$ have no common zero, thus by the Nullstellensatz there are $h_1, \dots, h_t \in K[x_1, \dots, x_{t-1}]$ such that

$$\sum_{i=1}^{t} h_i g_i = 1.$$

This is a contradiction, since the left side is in $\mathfrak{m}$ ($h_i \in R$!!!).

B is integrally closed in $QF(B)$: Since by the above A is integral over B and $x_1, \dots, x_t$ are algebraically independent over K, we have that $f_1, \dots, f_t$ are algebraically independent over K. Hence $B \cong A$ and thus is a UFD. The claim now follows since *every* UFD is integrally closed in its field of fractions.

Computing the rank: Let $\mathfrak{m} = (f_1, \dots, f_t) \triangleleft B = K[f_1, \dots, f_t]$, note that $\mathfrak{m}$ is maximal. Since A is a finitely generated B-module, $A_\mathfrak{m}$ is a finitely generated $B_\mathfrak{m}$-module, and therefore $A/\mathfrak{m}A \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$ is a finite dimensional $B/\mathfrak{m} \cong B_\mathfrak{m}/\mathfrak{m}B_\mathfrak{m} \cong K$-vector space.

If $\bar{x}_1, \dots, \bar{x}_s \in A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$ is a basis over K, then by Nakayama's Lemma we get that $x_1, \dots, x_s$ generate $A_\mathfrak{m}$ over $B_\mathfrak{m}$, and hence $x_1, \dots, x_s$ generate $QF(A)$ over $QF(B)$.

Thus $\dim_{QF(B)} QF(A) \leq \dim_K A/\mathfrak{m}A$. Then we have what we want since

$$A/\mathfrak{m}A \overset{\text{Claim 2}}{=} A/\cap\mathfrak{m}_\sigma \overset{\substack{\text{Chinese}\\ \text{Remainder}}}{\cong} \bigoplus_{\sigma \in S_t} A/\mathfrak{m}_\sigma \cong \bigoplus_{\sigma \in S_t} K,$$

where $\mathfrak{m}_\sigma := (x_1 - a_{1\sigma(1)}, \dots, x_t - a_{t\sigma(t)}) \triangleleft A$ is maximal.

So to conclude the proof of the Key Lemma we need to show the following.

Claim 2 *Let $\mathfrak{m}_\sigma = (x_1 - a_{1\sigma(1)}, \dots, x_t - a_{t\sigma(t)})$. Then*

$$\mathfrak{m}A = \bigcap_{\sigma \in S_t} \mathfrak{m}_\sigma.$$

Proof. $\boxed{\subseteq}$ Obvious.

$\boxed{\supseteq}$ Let

$$g_\sigma = \frac{f_1 \cdots f_t}{\prod_{i=1}^t (x_i - a_{i\sigma(i)})}.$$

The polynomials f_i, $i = 1, \dots t$ and g_σ, $\sigma \in S_t$ have no common zero, so there exists $h_i, h_\sigma \in K[x_1, \dots, x_t]$ such that $\sum f_i h_i + \sum g_\sigma h_\sigma = 1$.

Let $g \in \cap\mathfrak{m}_\sigma = \prod \mathfrak{m}_\sigma$, and let $g = \sum h_i f_i g + \sum h_\sigma g_\sigma g$. We know that $\sum h_i f_i g \in \mathfrak{m}A$, so $g_\sigma g \in \mathfrak{m}A$ (for $\forall \sigma \in S_t$) would imply $g \in \mathfrak{m}A$.

Fix a permutation $\sigma \in S_t$. We have $g = \sum g^*$, where a typical term $g^* = g' \cdot \prod_{\tau \in S_t} m_\tau$ with $g' \in A$ and $m_\tau \in \{x_1 - a_{1\tau(1)}, \dots, x_t - a_{t\tau(t)}\}$ for all $\tau \in S_t$. In particular let j be the index for which $m_\sigma = x_j - a_{j\sigma(j)}$. Then

$$f_j | g^* g_\sigma \quad \Rightarrow \quad g^* g_\sigma \in \mathfrak{m}A \quad \Rightarrow \quad g g_\sigma \in \mathfrak{m}A.$$

□

Hence we concluded the proof of the Key Lemma. □

2.3.2 The projective norm-graphs

Even denser $K_{3,3}$-free graphs — an elementary construction

The graph $H = H_{q,3}$ is defined as follows. The vertex set $V(H)$ is $\mathbb{F}_{q^2} \times \mathbb{F}_q^*$. Two distinct vertices (A, a) and $(B, b) \in V(H)$ are connected if and only if $N(A + B) = ab$, where $N(X) = X^{1+q}$ is the norm of $X \in \mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Of course $N(X) \in \mathbb{F}_q$ and it is clear that $|V(H)| = q^3 - q^2$. If $N(A+X) = ax$, then (A, a) and $X \neq -A$ determine x. Thus for any fixed $(A, a) \in V(H)$, there are exactly $q^2 - 1$ solutions (X, x) to $N(A + X) = ax$. This implies that, excluding possible loops, the degree of each vertex is at least $q^2 - 2 \geq n^{2/3}$.

We prove that H is $K_{3,3}$-free and hence provides an improvement (in the second order term) over Brown's construction for a dense $K_{3,3}$-free graph. (The Brown-graph has $\frac{1}{2}n^{5/3} - \frac{1}{2}n^{4/3}$ edges for infinitely many values of n.)

Theorem 2.10 *The graph $H = H_{q,3}$ contains no subgraph isomorphic to $K_{3,3}$. Thus there exists a constant C such that for every $n = q^3 - q^2$ where q is a prime power*

$$ex(n, K_{3,3}) \geq \frac{1}{2}n^{\frac{5}{3}} + \frac{1}{3}n^{\frac{4}{3}} + C.$$

Remark: At this point it is worthwhile to recall that the upper bound of Füredi (Theorem 2.8) is

$$ex(n, K_{3,3}) \leq \frac{1}{2}n^{\frac{5}{3}} + n^{\frac{4}{3}} + \frac{1}{2}n.$$

Proof. The statement of Theorem 2.10 is a direct consequence of the following.
If $(D_1, d_1), (D_2, d_2), (D_3, d_3)$ are distinct elements of $V(H)$, then the system of equations

$$\begin{aligned} N(X + D_1) &= xd_1 \\ N(X + D_2) &= xd_2 \\ N(X + D_3) &= xd_3 \end{aligned} \tag{2.7}$$

has at most two solutions $(X, x) \in \mathbb{F}_{q^2} \times \mathbb{F}_q^*$.

Observe that if the system has at least one common solution (X, x), then

(i) $X \neq -D_i$ for any $i = 1, 2, 3$ and

(ii) $D_i \neq D_j$ if $i \neq j$.

The latter is true, because if $D_i = D_j$, then the presence of a common neighbor implies $d_i = d_j$.

Because of *(i)* we can divide the first two equations by the last one and get rid of x. The norm is a multiplicative function, so we obtain

$$N\left(\frac{X + D_i}{X + D_3}\right) = \frac{d_i}{d_3},$$

for $i = 1, 2$.

We can divide each equation by $N(D_i - D_3)$, since these are nonzero by *(ii)*. Then we can substitute $Y = 1/(X + D_3), A_i = 1/(D_i - D_3)$ and $b_i = d_i/(d_3 N(D_i - D_3))$ and obtain the following two equations:

$$\begin{array}{rcccl} N(Y + A_1) & = & (Y + A_1)(Y^q + A_1^q) & = & b_1 \\ N(Y + A_2) & = & (Y + A_2)(Y^q + A_2^q) & = & b_2. \end{array} \tag{2.8}$$

Here we used the fact that $(A + B)^q = A^q + B^q$ for all A, B in $\mathbb{F}_{q^2}$.

We need the following simple special case of our Key Lemma from the previous section.

Lemma 2.10.1 *Let K be a field and $a_{ij}, b_i \in K$ for $1 \leq i, j \leq 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations*

$$\begin{array}{rcl} (x_1 - a_{11})(x_2 - a_{12}) & = & b_1, \\ (x_1 - a_{21})(x_2 - a_{22}) & = & b_2 \end{array} \tag{2.9}$$

has at most two solutions $(x_1, x_2) \in K^2$.

Although this is a special case of the Key Lemma (for $t = 2$) we include an elementary proof, which does not use any commutative algebra. We note that even for $t = 3$, we do not know of any proof of the Key Lemma which is simpler than the one in the previous section.

Proof. Subtracting the first equation from the second we get

$$(a_{11} - a_{21})x_2 + (a_{12} - a_{22})x_1 + a_{21}a_{22} - a_{11}a_{12} = b_2 - b_1.$$

Here we can express x_1 in terms of a linear function of x_2, since $a_{12} \neq a_{22}$. Substituting this back into one of the two equations of (2.9) we obtain a quadratic equation in x_2 with a non-zero leading coefficient (since $a_{11} \neq a_{21}$). This equation has at most two solutions in x_2 and each one determines x_1 uniquely. □

In order to finish the proof of Theorem 2.10 we apply Lemma 2.10.1 with $x_1 = Y, x_2 = Y^q$, $a_{11} = -A_1, a_{12} = -A_1^q$, and $a_{21} = -A_2, a_{22} = -A_2^q$. The conditions of the lemma hold since $-A_1^q = a_{12} = a_{22} = -A_2^q$ would mean $A_1 = A_2$, which is impossible by *(ii)*. Hence the system of equations (2.8) has at most two solutions Y. These solutions are in one-to-one correspondence with the solutions (X, x) of the equations (2.7), so Theorem 2.10 is proved. □

Exercise 2.14 *The k*-color Ramsey number *$R_k(G)$ is the smallest integer m, such that no matter how the edges of K_m are colored with k colors, there exists a monochromatic copy of G.*

Show that $R_k(K_{3,3}) = (1 + o(1))k^3$.

(Hint: *For the lower bound use the projective norm-graphs and the Key Lemma.)*

The general projective norm-graphs

The proof of Theorem 2.10 in the previous subsection is completely elementary. In order to prove the properties of the projective norm-graphs for $t > 3$ we need the Key Lemma for arbitrary t.

Let us define the projective norm-graph $H = H_{q,t}$ for any $t > 2$. The vertex set of H is $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$. Two distinct vertices (A, a) and $(B, b) \in V(G)$ are adjacent if and only if $N(A + B) = ab$, where the norm is understood over $\mathbb{F}_q$, that is, $N(x) = x^{1+q+\cdots+q^{t-2}}$. Note that $|V(H)| = q^t - q^{t-1}$. If (A, a) and (B, b) are adjacent, then (A, a) and $B \neq -A$ determine b. Thus H is almost regular with possible degrees $q^{t-1} - 1$ and $q^{t-1} - 2$.

Theorem 2.11 *The graph $H = H_{q,t}$ contains no subgraph isomorphic to $K_{t,(t-1)!+1}$.*

Proof. The proof is a straightforward generalization of the proof of Theorem 2.10 with the remark that we need to use Lemma 2.8.1 (for $t - 1$ equations) instead of Lemma 2.10.1. □

Therefore, the following improvement over the previous section holds.

Corollary 2.12 *For every fixed $t \geq 2$ and $s \geq (t-1)! + 1$ we have*

$$ex(n, K_{t,s}) \geq \frac{1}{2}n^{2-\frac{1}{t}} - O(n^{2-\frac{1}{t}-c}),$$

where $c > 0$ is an absolute constant.

The improvement is most visible for small values of t, for example:

Corollary 2.13

$$ex(n, K_{4,7}) = \Theta(n^{7/4}).$$

Open Problem. The value of $ex(n, K_{4,4})$ is wide open. The conjecture is of course $\Theta\left(n^{7/4}\right)$, but we can't even separate it from $ex(n, K_{3,3})$, that is, we don't know whether

$$\lim_{n\to\infty} \frac{ex(n, K_{4,4})}{n^{5/3}} \to \infty.$$

Settling this would already be a major advance, even though most experts would be willing to bet a significant amount of money on that it is true.

Füredi's construction of dense $K_{2,s}$-free graphs (Theorem 2.3) showed that

$$\lim_{s\to\infty}(\liminf_{n\to\infty} ex(n, K_{2,s})n^{-3/2}) = \lim_{s\to\infty} \frac{1}{2}\sqrt{s-1} = \infty.$$

Exercise 2.15 *Use the projective norm-graphs together with Füredi's idea (which improves the $K_{2,2}$-free Construction 2 to a $K_{2,s}$-free construction) to give a $K_{3,s}$-free construction whose number of edges comes within a factor $\sqrt[3]{2} + o(1)$ of the KST upper bound of $ex(n, K_{3,s})$ for every $s \geq 3$ of the form $s = 2r^2 + 1$, $r \in \mathbb{Z}$. (The $o(1)$ above is understood as $s \to \infty$.)*
More generally, prove that for any fixed t

$$\lim_{s\to\infty}(\liminf_{n\to\infty} ex(n, K_{t,s})n^{-(2-1/t)}) = \infty.$$