Constructive Combinatorics So 2015

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Exercise Sheet 1

Due date: Apr 20th, 5:00 PM, tutor box of Shagnik Das
Late submissions will be frowned upon most sternly.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1. In this exercise you will provide another proof of Turán's theorem.

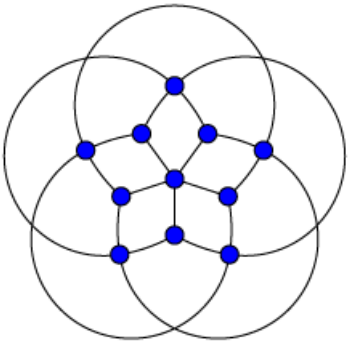
(i) Suppose G is a K_k-free graph, and let u and v be non-adjacent vertices of G. Let G' be the graph obtained by replacing the neighbourhood of v with the neighbourhood of u; that is, for $w \notin \{u, v\}$, w is adjacent to v if and only if it is adjacent to u, and all other edges are unchanged. Show that G' is also K_k-free.

(ii) Deduce that if there is an n-vertex K_k-free graph with m edges, there is a complete $(k-1)$-partite n-vertex K_k-free graph with at least m edges.

(iii) Deduce Turán's theorem; that is, show that the Turán graph $T_{n,k-1}$ is optimal.

Exercise 2. Use the method of the previous exercise to prove the following: for any k, the n-vertex K_k-free graph with the maximum number of copies of K_3 is the Turán graph $T_{n,k-1}$. [Remark: one can generalise this by replacing 'K_3' with 'K_t' for any t.]

Exercise 3. Find the n-vertex graph with as many edges as possible that does not contain the graph below as a subgraph.



Exercise 4. Your TV remote requires two working batteries to function. Opening the drawer in which you keep your batteries, you find eight batteries. You remember that four of them work and four of them do not, but there is no way of telling them apart without testing them in the remote. How quickly can you guarantee to get your remote working?