Constructive Combinatorics So 2015
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Exercise Sheet 2

Due date: May 4th, 5:00 PM, tutor box of Shagnik Das
Late submissions may be transformed into paper aeroplanes.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 For the octahedron, $K_{2,2,2}$, show that for all $n \geq 3$ we have the *strict* inequality $\mathrm{ex}(n, K_{2,2,2}) > e(T_{n,2})$. How large an n-vertex octahedron-free graph can you find?

Exercise 2 Show that for any tree T with t edges, $\frac{(t-1)n}{2} - o(n) \leq \mathrm{ex}(n, T) \leq (t-1)n$. In the special case of the star graph, $T = K_{1,t}$, show that the lower bound is correct.

Exercise 3 A *projective plane* Π is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is a set of elements called *points*, $\mathcal{L} \subseteq 2^{\mathcal{P}}$ is a family of subsets of points called *lines*, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an *incidence relation* between points and lines such that the following holds:

(i) $\forall \ell_1, \ell_2 \in \mathcal{L}, \ell_1 \neq \ell_2,\ \exists! p \in \mathcal{P}\ : (p, \ell_1), (p, \ell_2) \in \mathcal{I}$
(every pair of distinct lines is incident to a unique common point),

(ii) $\forall p_1, p_2 \in \mathcal{P}, p_1 \neq p_2,\ \exists! \ell \in \mathcal{L}\ : (p_1, \ell), (p_2, \ell) \in \mathcal{I}$
(every pair of distinct points is incident to a unique common line),

(iii) there are four points such that no three of them are incident to a single line (non-degeneracy).

A projective plane is called finite if $\mathcal{P}$ is a finite set. Prove that for any finite projective plane there is an integer $m \geq 2$ such that every point (line, respectively) of Π is incident to exactly $m+1$ lines (points, respectively) and that $|\mathcal{P}| = m^2 + m + 1 = |\mathcal{L}|$. (The integer m is called the *order* of the finite projective plane.)

Exercise 4 For a prime p and every $\alpha \in \mathbb{F}_p$, determine the number of solutions in $\mathbb{F}_p$ to the equation $x^2 + y^2 = \alpha$.

[Hint (to be read backwards): tI yam pleh ot tnemirepxe htiw llams seulav fo p dna α.]

Exercise 5 Given a bipartite graph H, define $\mathrm{ex}(n, m, H)$ as the maximum number of edges in an H-free bipartite graph with partite sets of size n and m respectively. Show that $\mathrm{ex}(q^2 + q + 1, q^2 + q + 1, K_{2,2}) = (q^2 + q + 1)(q + 1)$ for every prime power q.

Exercise 6 Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if G is an n-vertex graph with at least $\left(\frac{1}{4} + \varepsilon\right) n^2$ edges, then G contains at least δn^3 triangles.