Constructive Combinatorics So 2015
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Exercise Sheet 3

Due date: May 18th, 5:00 PM, tutor box of Shagnik Das
Late submissions will not receive a warm welcome.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if G is an n-vertex graph with at least $\left(\frac{1}{4} + \varepsilon\right) n^2$ edges, then G contains at least δn^3 triangles.

[Hint (to be read backwards): A elgnairt si tsuj na egde dna a nommoc ruobhgien fo sti stniopdne. naC uoy dnif a doog rewol dnuob rof eht rebmun fo nommoc sruobhgien a riap fo secitrev sah?]

Exercise 2 This exercise shows that a straightforward generalisation of the $K_{2,s}$-free constructions we have seen fails to provide $K_{3,s}$-free graphs.

A *hyperplane* in $\mathbb{F}_q^3$ is an affine subspace of dimension two. We construct a bipartite graph G, with vertex classes $\mathcal{P} = \mathbb{F}_q^3$ and $\mathcal{H}$, the set of hyperplanes in $\mathbb{F}_q^3$. A pair $(p, H) \in \mathcal{P} \times \mathcal{H}$ forms an edge if and only if $p \in H$. (This is the generalisation of Construction 0 for $K_{2,2}$-free graphs.)

(i) How many vertices and edges does G have?

(ii) Show that if we choose H_1, H_2, and H_3 uniformly at random from $\mathcal{H}$, the probability that $|H_1 \cap H_2 \cap H_3| = 1$ tends to 1 as $q \to \infty$.

(iii) Despite this fact, show that any choice of $10q^2$ hyperplanes contains three planes that intersect in at least q points, thus inducing a $K_{3,q}$ in G.

Exercise 3 The α-Brown graph G has vertices $\mathbb{F}_p^3$, with (x_1, x_2, x_3) adjacent to (y_1, y_2, y_3) if and only if $\sum_{i=1}^3 (x_i - y_i)^2 = \alpha$. For each value of α, determine exactly how many edges the α-Brown graph has.

Deduce from this that the projective polarity graph (c.f. Construction 3 of a $K_{2,2}$-free graph), with vertices the points $[x_0, x_1, x_2]$ of the projective plane over $\mathbb{F}_q$ and an edge between points $\mathbf{x}$ and $\mathbf{y}$ if and only if $x_0y_0 + x_1y_1 + x_2y_2 = 0$, has exactly $q + 1$ vertices of degree q, with the remaining q^2 vertices having degree $q + 1$.

Exercise 4 Show that if $-\alpha \in QR(p)$, then the α-Brown graph contains not only a $K_{3,3}$, but also a $K_{n^{1/3},n^{1/3}}$. Can you also find a $K_{n^{1/3},n^{2/3}}$?

Exercise 5 Prove that in the α-Brown graph, when $-\alpha \notin QR(p)$, roughly half of the triples of vertices have two common neighbours while the other half have none. Even further, explicitly describe the triples without common neighbours.