Constructive Combinatorics So 2015
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Exercise Sheet 4

Due date: June 1st 5:00 PM, tutor box of Shagnik Das
Late submissions may mysteriously disappear.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 The k-colour Ramsey number $R_k(G)$ is the smallest integer n such that in every colouring of the edges of K_n with k colours, there is a monochromatic copy of G.

Show that $R_k(K_{3,3}) = (1 + o(1))k^3$.

[Hint (to be read backwards): roF eht rewol dnuob, esu eht evitcejorp shparg-mron dna eht yeK ammeL.]

Exercise 2 Prove the even-girth Moore bound: that for any n-vertex d-regular graph G with girth $g(G) \geq 2k$,

$$n \geq 2 \sum_{i=0}^{k-1} (d-1)^i.$$

Can you extend this to irregular graphs?

Exercise 3 Let X and Y be discrete random variables. Let $\mathbb{E}_x$ denote the expectation when x is distributed as X, and $\mathbb{E}_y$ the expectation when y is distributed as Y. We define the *entropy* of X as $H(X) = \mathbb{E}_x[-\log_2 \mathbb{P}(X = x)]$,[1] where we set $z \log_2 z = 0$ for $z = 0$. We can also define the *joint entropy* $H(X, Y) = \mathbb{E}_{x,y}[-\log_2 \mathbb{P}(X = x, Y = y)]$ and the *conditional entropy* $H(X|Y) = \mathbb{E}_y[\mathbb{E}_{x|y}[-\log_2 \mathbb{P}(X = x|Y = y)]]$. Prove that $H(X, Y) = H(Y) + H(X|Y)$.

Exercise 4 Let K be a cycle of length n with a chord (an edge between two non-neighbouring vertices on the cycle), and let $\chi : V(K) \to \{\text{red}, \text{blue}\}$ be a non-constant two-colouring that is *not* proper. Show that for every $\ell < n$ there exists a path of length ℓ in K whose endpoints are coloured differently.

[Hint (to be read backwards): fI siht si ton eurt, redisnoc eht tsetrohs gnissim htgnel dna wohs taht eht gniruoloc tsum eb cidoirep dnuora eht elcyc.]

[1] By definition of $\mathbb{E}_x$, this is equivalent to $-\sum_x \mathbb{P}(X = x) \log_2 \mathbb{P}(X = x)$.

Exercise 5 Combine Füredi's idea for the construction of $K_{2,s}$-free graphs with the projective norm-graphs to give a $K_{3,s}$-free constructions whose number of edges is within a $(\sqrt[3]{2}+o(1))$-factor of the Kővári–Sós–Turán bound for every $s = 2r^2+1$, where $r \in \mathbb{N}$. (The $o(1)$ term should be understood to go to zero as s tends to infinity.)