Constructive Combinatorics | So 2015

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Exercise Sheet 6

Due date: June 29th 5:00 PM, tutor box of Shagnik Das
Late submissions may face an interminable wait for judgement.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Let G and H be two graphs, and let $G \otimes H$ be their Abbott product.

(i) Verify that $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$.

We first constructed superpolynomially large Ramsey graphs by taking the Abbott product $\mathcal{G}_{n_0}$ of all labelled graphs on n_0 vertices.

(ii) Recalling the bounds we obtained on the number of vertices and the clique and independence numbers of $\mathcal{G}_n$, verify that this construction shows $R(k,k) \geq k^{\Omega\left(\frac{\log\log\log k}{\log\log\log\log k}\right)}$.

Exercise 2 The Paley graph P_q is defined on vertices $\mathbb{F}_q$, where q is a prime power congruent to 1 modulo 4, with $xy \in E$ if and only if $x - y$ is a quadratic residue in $\mathbb{F}_q$.

(i) Show that P_q is isomorphic to its complement.

(ii) Show that P_q is edge-transitive; that is, for every pair of edges xy and uv in E, there is an isomorphism of P_q mapping x to u and y to v.

(iii) By considering P_{17}, show that $R(4,4) = 18$.

Exercise 3 Suppose that addition and multiplication in $\mathbb{F}_q$ can be carried out in constant time. Show that P_q is strongly explicit; that is, there is some constant C such that one can decide if two given vertices x and y are adjacent in $O(\log^C(q))$ time. How long does it take to determine the entire graph P_q?

Exercise 4 Prove that $\alpha(P_q) \leq \sqrt{p} + 2$. In the special case $q = p^2$ for an odd prime p, show $\alpha(P_q) \geq \sqrt{q}$.

[Hint (to be read backwards): oT evorp eht reppu dnuob, terpretni eht seulavnegie fo eht yelaP hparg (hcihw si a yelyaC hparg) sa naissuaG smus, dna neht dnuob meht yletairporppa.]

Exercise 5

(i) Calculate all the eigenvalues of K_n, and find a basis of eigenvectors.

(ii) Show that if G is an (n, d, λ)-graph with $d \leq n/2$, then $\sqrt{d/2} \leq \lambda \leq d$.

[Hint (to be read backwards): roF eht reppu dnuob, redisnoc eht tsegral etanidrooc ni na rotcevnegie. roF eht rewol dnuob, yrt gnitnuoc desolc sklaw fo htgnel owt ni owt tnereffid syaw. roF eht hctaW![1]]

[1] sihT tsal trap t'nsi yllaer lufpleh.