Constructive Combinatorics So 2015
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Exercise Sheet 7

Due date: July 13th 5:00 PM, tutor box of Shagnik Das
Late submissions may get lost in some airport.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 The construction of Codenotti–Pudlak–Resta we saw gave a constructive lower bound of $R(3,k)$ of order $k^{4/3}$. In this exercise, we seek to improve this bound by making the following changes to the construction G:

- Instead of starting with the girth-8 Benson graph, take the underlying graph B to be the point/line incidence graph of the projective plane.
- As before, the edges of the underlying graph will be the vertices of our construction; that is, $V(G) = E(B)$.
- Fix an arbitrary ordering $\prec$ on the set P of points of the projective planes.
- Put an edge between vertices $p_1\ell_1$ and $p_2\ell_2$ of G if $p_1 \prec p_2$, $\ell_1 \neq \ell_2$, and $p_1\ell_2 \in E(B)$.

Show that this graph gives a constructive lower bound of order $k^{3/2}$ for $R(3,k)$.

Exercise 2 Recall that an *s-uniform sunflower with r petals* is a collection of sets $F_1, F_2, \ldots, F_r$, each of size s, such that $F_i \cap F_j = \cap_{\ell=1}^{r} F_\ell$ for every $1 \leq i < j \leq r$.

(i) Show that any family $\mathcal{F}$ of more than $s!(r-1)^s$ sets, each of size s, must contain a sunflower with r petals.

(ii) Build an s-uniform family with at least 2^s sets not containing a sunflower with 3 petals.

[Hint (to be read backwards): nI trap (i), esu noitcudni no eht ytimrofinu s fo eht ylimaf. rebmemeR taht eht nommoc noitcesretni nac eb ytpme!]

Exercise 3 Show that the projective norm-graphs provide a constructive lower bound of $\Omega\left(k^{4/3}\right)$ for $R(C_4, K_k)$.[1]

[1] The best-known lower bound, obtained by the Local Lemma, is of order $(k/\log k)^{3/2}$.

Exercise 4 The following two exercises concern multicoloured Ramsey numbers.

(i) Provide a (reasonable[2]) definition of the multicoloured Ramsey number

$$R_r(3) := R(\underbrace{3, 3, \ldots, 3}_{r}).$$

(ii) Show that $R_r(3)$ is finite by proving $R_r(3) \leq (R_{r-1}(3) - 1)\, r + 2$.

(iii) Deduce the upper bound $R_r(3) \leq \lfloor e \cdot r! \rfloor + 1$.

Exercise 5 Prove that for every $k \geq 1$, $R(\underbrace{3, 3, \ldots, 3}_{k}, m) \leq C_k m^{k+1}$, where C_k is a constant independent of m.

[2]Your definition should agree with $R(3, 3)$ when $r = 2$, and allow for non-trivial theorems.