Erdős-Simonovits-Stone Theorem

**Theorem.** (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r - 1}\right) \binom{n}{2} + o(n^2).$$

**Corollary.** (Erdős-Simonovits, 1966) For any graph $H$,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2).$$

**Corollaries of the Corollary.**

$$ex(n, \text{octahedron}) = \frac{n^2}{4} + o(n^2)$$

$$ex(n, \text{dodecahedron}) = \frac{n^2}{4} + o(n^2)$$

$$ex(n, \text{icosahedron}) = \frac{n^2}{3} + o(n^2)$$

$$ex(n, \text{cube}) = o(n^2)$$
Proof of the Erdős-Simonovits Corollary

**Theorem.** (Erdős-Stone, 1946) For arbitrary fixed integers \( r \geq 2 \) and \( t \geq 1 \)

\[
\text{ex}(n, T_{rt, r}) = \left(1 - \frac{1}{r - 1}\right) \binom{n}{2} + o(n^2).
\]

**Corollary.** (Erdős-Simonovits, 1966) For any graph \( H \),

\[
\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2).
\]

**Proof of the Corollary.** Let \( r = \chi(H) \).

- \( \chi(T_{n, r-1}) < \chi(H) \), so \( e(T_{n, r-1}) \leq \text{ex}(n, H) \).

- \( T_{r\alpha, r} \supseteq H \), so \( \text{ex}(n, T_{r\alpha, r}) \geq \text{ex}(n, H) \), where \( \alpha \) is a constant depending on \( H \); say \( \alpha = \alpha(H) \).
Proof of the Erdős-Stone Thm

Erdős-Stone Theorem. (Understanding precisely what it actually says) For any $\epsilon > 0$ and integers $r \geq 2, t \geq 1$ there exists an integer $M = M(r, t, \epsilon)$, such that any graph $G$ on $n \geq M$ vertices with more than $$\left(1 - \frac{1}{r-1} + \epsilon \right) \binom{n}{2}$$ edges contains $T_{rt,r}$.

We derive this through the following statement.

Seemingly Weaker Theorem. For any $\epsilon > 0$ and integers $r \geq 2, t \geq 1$ there exists an integer $N = N(r, t, \epsilon)$, such that any graph $G$ on $n \geq N$ vertices and with $\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon \right) n$ contains $T_{rt,r}$.

Note that w.l.o.g. $\epsilon < \frac{1}{r-1}$. 
**Derivation of the Erdős-Stone Theorem from the See-mingly Weaker Theorem.**

Let $G$ be a graph on $n \geq M(r, t, \varepsilon)^*$ vertices with more than $\left(1 - \frac{1}{r-1} + \varepsilon\right) \binom{n}{2}$ edges. Recursively delete vertices which are adjacent to less than $\left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)$-fraction of the remaining vertices.

What is the number $n'$ of vertices we are left with?

We deleted at most 
$$\sum_{j=n'+1}^{n} j \left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)$$ 
edges. So

$$e(G) \leq \left(\binom{n+1}{2} - \binom{n'+1}{2}\right) \left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right) + \binom{n'}{2}.$$ 

This implies

$$\frac{\varepsilon}{2} \binom{n}{2} - n \leq \left(\frac{1}{r-1} - \frac{\varepsilon}{2}\right) \binom{n'}{2} - n'.$$

We choose $M(r, t, \varepsilon)$ such that $n \geq M(r, t, \varepsilon)$ implies $n' \geq N(r, t, \varepsilon/2)$.

*At this point we don’t know $M(r, t, \varepsilon)$ yet!!! We’ll define it in the proof through $N(r, t, \varepsilon/2)$. (which is known!)*
Proof of the Seemingly Weaker Theorem.
Induction on $r$.
For $r = 2$ the claim is true provided \( \left( \frac{en}{t} \right)^n > t - 1 \), which is certainly true from some threshold \( N(2, t, \epsilon) \).

Let $r \geq 2$ and $G$ be a graph on $n \geq N(r + 1, t, \epsilon)^*$ vertices with $\delta(G) \geq \left( 1 - \frac{1}{r} + \epsilon \right) n$.
We would like to find a $T_{(r+1)t,r+1}$ in $G$.

Let $s = \left\lceil \frac{t}{\epsilon} \right\rceil$. By the induction hypothesis\( ^\dagger \) there is a $T_{rs,r}$ in $G$ with vertex-set $A_1 \cup \ldots \cup A_r$, where $|A_1| = \ldots = |A_r| = s$.

\[ U = V(G) \setminus (A_1 \cup \ldots \cup A_r). \]

\[ W = \{ w \in U : |N(w) \cap A_i| \geq t, i = 1, \ldots, r \} \]
is the set of vertices eligible to extend some part of $A_1, \ldots, A_r$ into a $T_{(r+1)t,r+1}$.

\( ^* \)Again, we don’t know \( N(r + 1, t, \epsilon) \) yet.
\( ^\dagger \)Here we assume \( N(r + 1, t, \epsilon) \geq N(r, s, \epsilon) \).
Double-count the number of edges missing between \( U \) and \( A_1 \cup \ldots \cup A_r \). They are

- at least \((|U| - |W|)(s - t) \ (\approx (s - t)n\text{ if } W\text{ is small})\)
- at most \(rs \left(\frac{1}{r} - \epsilon\right)n \ (\approx (s - rt)n, \text{ if } W\text{ is small})\)

From this we have

\[
|W| \geq \frac{(r - 1)\epsilon}{1 - \epsilon}n - rs
\]

Thus if \( n \) is large enough* then

\[
|W| > \binom{s}{t}^r (t - 1).
\]

So we can select \( t \) vertices from \( W \), which are adjacent to the same \( t \) vertices in each \( A_i \).

*If \( N(r + 1, t, \epsilon) > \left(\binom{s}{t}^r (t - 1) + rs\right) \frac{1 - \epsilon}{(r - 1)\epsilon} \)