Structural Graph Theory Winter 2016-17
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Exercise Sheet 4

Due date: 08:00am, November 30th.

You should try to solve all of the exercises below. In order to get a 'plus' for this sheet you should submit correct solutions for exercises worth at least half of the total (4) number of points. You can submit solutions in print (my office) or by email. The solutions will be graded.

Exercise 0 [`1 points`] Let $\delta(G)$ be a minimum degree of a vertex in G. Prove that $\mathrm{tw}(G) \geqslant \delta(G)$.

Exercise 1 [`1 points`] The *helicopter game* is a variant of the Cops and Robber game. Here, the cop player starts placing her k cops on vertices of G. Next, the robber chooses a vertex for his piece. Then they play in rounds. In each round the Cop player chooses one of her cops and removes him from the graph (puts him into a helicopter). She also announces where this cop will reappear (so on which vertex the helicopter lands). With this information the robber can move in G, many times, from his current vertex to an adjacent one and so on. Clearly, if the robber enters a vertex with a cop standing there he loses. Finally, the robber settles on some chosen vertex and the helicopter lands on the vertex announced before.

This finishes a single round. The *helicopter number* of a graph G is a minimum number d such that the Cop player has a winning strategy playing with d cops. Prove that the helicopter number of G is equal to the tree-width of G plus one.

Exercise 2 [`1 points`] Let $n \times n$ be a grid with n rows and n columns. Show that $\mathrm{tw}(n \times n) = n$.

Exercise 3 [`1 points`] The *path-width* $\mathrm{pw}(G)$ of a graph G is the least width of a path-decomposition of G. A *path-decomposition* of G is a tree-decomposition $(T, \mathcal{V})$ where T is a path.

Do trees have unbounded path-width?

Exercise 4 [`1 points`] A set X is *externally k-connected* in a graph G, if $|X| \geqslant k$ and for every $Y, Z \subseteq X$ with $|Y| = |Z| \leqslant k$ there are $|Y|$ pairwise disjoint Y-Z paths in G without inner vertex and edge in X.

Prove that any horizontal line in a grid $n \times n$ is externally n-connected.

Exercise 5 [`1 points`] Show that the tree-width of a graph is large if and only if it contains a large externally k-connected set of vertices, with large k. For example, show that graphs of tree-width $< k$ contain no externally $(k+1)$-connected set of $3k$ vertices, and that graphs containing no externally $(k+1)$-connected set of $3k$ vertices have tree-width $< 4k$.

Exercies 6 [`1 points`] Find an $\mathbb{N} \to \mathbb{N}^2$ function $k \to (h, \ell)$ such that every graph with an externally ℓ-connected set of h vertices contains a bramble of order at least k. Note that only this implies a weaker version of the Treewidth Duality Theorem that, given k, every graph of large enough tree-width contains a bramble of order at least k. (Clearly, the idea for this exercise is to not solve it using the Tree-width Duality Theorem but to come up with an independent argument giving somewhat a weaker version of the duality.)