Structural Graph Theory
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Winter 2016-17

Exercise Sheet 6

Due date: 08:00am, December 14th.

You should try to solve all of the exercises below. In order to get a 'plus' for this sheet you should submit correct solutions for exercises worth at least half of the total (4) number of points. You can submit solutions in print (my office) or by email. The solutions will be graded.

Exercise 1 [`2 points`] Recall that the *circumference* of a graph is the length of a longest cycle in a graph. (In other words, G has circumference ℓ ($\ell > 0$), if ℓ is the smallest integer such that $C_{\ell+1} \not\prec G$.)

Show that if a 2-connected graph has circumference k ($k > 0$), then its path-width is at most $O(k^2)$.

Exercise 2 [`2 points`] A *transaction* of a sequence $(v_1, \ldots, v_n)$ of vertices is a set of disjoint paths from an initial segment $\{v_1, \ldots, v_i\}$ to the rest, $\{v_{i+1}, \ldots, v_n\}$.

Given $k \in \mathbb{N}$ and a sequence $v_1, \ldots, v_n$ of all vertices of G, show that G has a path-decomposition $(V_1, \ldots, V_n)$ with $|V_i \cap V_{i+1}| \leqslant k$ and $v_i \in V_i$ for all i, if and only if G contains no transaction $\mathcal{P}$ of $(v_1, \ldots, v_n)$ of order $|\mathcal{P}| > k$.

Exercise 3 [`2 points`] Prove that for every forest F, every graph G with $\mathrm{pw}(G) \geqslant |V(F)| - 1$, we have $F \prec G$. (You can try to use a kind of Path-width Duality Theorem, see the next exercise, or look up the paper *On the exclusion of forest minors: a short proof of the path-width theorem* by Reinhard Diestel)

Exercise 4 [`2 points`] A pair (A, B) of subsets of $V(G)$ is a *separation* of G, if there is no edge between $A - B$ and $B - A$. The *order* of this separation is $|A \cap B|$.

A family $\mathcal{S}$ of separations of G is a *stoppage* of order k if:

(i) for each separation (A, B) of order at most k in G either $(A, B) \in \mathcal{S}$, (x)or $(B, A) \in \mathcal{S}$, but not both;

(ii) if $(A_1, B_1), (A_2, B_2) \in \mathcal{S}$, then $G[A_1] \cup G[A_2] \neq G$ ($G[A]$ means the subgraph that G induces on a set A).

When $(A, B) \in \mathcal{S}$ you can think of A as the 'small side' and B as the 'big side' of the separation.

Prove that: $\mathrm{pw}(G) \geqslant k$ if and only if G has a stoppage of order k.