Structural Graph Theory
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Winter 2016-17

# Exercise Sheet 7

**Due date: 08:00am, January 18.**

This set contains somewhat easier than usually exercises. Though I believe that it is going to help us to understand and work with tangles. I ask you to solve this time more than half ...
You should try to solve all of the exercises below. In order to get a 'plus' for this sheet you should submit correct solutions for exercises worth at least half of the total (14) number of points. You can submit solutions in print (my office) or by email. The solutions will be graded.

**Exercise 1** [`1 points`] Show that the following graphs have treewidth at least 4.

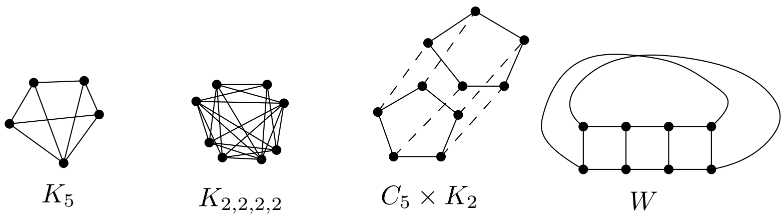


Figure 1: The Kuratowski set of graphs for the class of graphs with treewidth $< 4$.

**Exercise 2** [`1 points`] Show that for every graph $G$

$$\mathrm{bw}(G) \leqslant \mathrm{tw}(G) + 1 \leqslant \left\lceil \frac{3}{2}\,\mathrm{bw}(G) \right\rceil .$$

**Exercise 3** [`1 points`] Prove that, if $\mathrm{bw}(G) > k \geqslant 2$, then $G$ has a highly connected set of size $k$. (This was sketched during a lecture.)

**Exercise 4** [`1 points`] Submodularity inequalities for separations.
Prove that if $(H_1, H_2)$ and $(G_1, G_2)$ are separations, then $(H_1 \cap G_1, H_2 \cup G_2)$ and $(H_1 \cup G_1, H_2 \cap G_2)$ are both separations and

$$\mathrm{ord}(H_1 \cap G_1, H_2 \cup G_2) + \mathrm{ord}(H_1 \cup G_1, H_2 \cap G_2) \leqslant \mathrm{ord}(H_1, H_2) + \mathrm{ord}(G_1, G_2).$$

**Exercise 5** [`1 points`] For a graph $G$ and a clique $X \subseteq G$, let $\mathcal{T}_k$ be the set of all separations $(A, B)$ of order $< k$ such that $X \subseteq V(B)$. Show that, if $k < \frac{2}{3}|X| + 1$, then $\mathcal{T}_k$ is a tangle of order $k$.

**Exercise 6** [`3 points`] Let $G$ be a $(k \times k)$-grid. Let $\mathcal{T}$ be he set of all separations $(A, B)$ of $G$ of order at most $k - 1$ such that $B$ contains some row of the grid. Prove that $\mathcal{T}$ is a tangle of order $k$.

**Exercise 7** [`1 points`] Let $\mathcal{T}$ be a tangle of order $k$ of $G$. Prove that

(i) If $(A, B)$ is a separation of $G$ with $V(A) < k$, then $(A, B) \in \mathcal{T}$;

(ii) If $(A, B) \in \mathcal{T}$ and $(A', B')$ is a separation of $G$ of order $< k$ such that $B'B$, then $(A', B') \in \mathcal{T}$;

(iii) If $(A, B), (A', B') \in \mathcal{T}$ and $\mathrm{ord}(A \cup A', B \cap B') < k$, then $(A \cup A', B \cap B') \in \mathcal{T}$.

**Exercise 8** [`1 points`] Prove that, if $G$ has a tangle of order $k$, then $\mathrm{bw}(G) \geqslant k$. (This was also at last lecture but we left for exercises some issues with isolated vertices.)

**Exercise 9** [`1 points`] Given a tangle $\mathcal{T}$ of order $k$ of $G$, we say that $I \subseteq V(G)$ is $\mathcal{T}$-independent if $|I| = \mathcal{K}_{\mathcal{T}}(I)$. Prove the following properties of $\mathcal{T}$-independent sets (You can and should derive it from the rank axioms properties of $\mathcal{K}_{\mathcal{T}}$):

(i) $\emptyset$ is $\mathcal{T}$-independent;

(ii) subsets of independent sets are $\mathcal{T}$-independent;

(iii) for each $X \subseteq V(G)$, the maximal ($\subseteq$-wise) $\mathcal{T}$-independent sets of $X$ all have size $\mathcal{K}_{\mathcal{T}}(X)$.

**Exercise 10** [`3 points`] Prove that: A graph $G$ has a tangle of order $k$ if and only if there is a family $\mathcal{C}$ of connected subgraphs of $G$ that touches triplewise and has no vertex cover of cardinality $< k$.