Discrete Mathematics III — So 2016

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Exercise Sheet 3

Due date: 12:30, May 18th, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Let $\{(A_i, B_i,\ 1 \le i \le h)\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = l$ for all i, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \ne \emptyset$ for all $i \ne j$. Prove that $h \le \dfrac{(k+l)^{k+l}}{k^k l^l}$.

Exercise 2 Suppose $p > n > 10m^2$, with p prime, and let $0 < a_1 < a_2 < \cdots < a_m < p$ be integers. Prove that there is an integer x, $0 < x < p$ for which the m numbers

$$(xa_i \bmod p) \bmod n, \qquad 1 \le i \le m$$

are pairwise distinct.

Exercise 3 Prove that there is a constant $c > 0$ such that for every $n \ge 4$ the following holds: For every undirected complete graph K on n vertices whose edges are colored red and blue, the number of alternating Hamiltonian cycles in K (i.e., properly edge-colored cycles of length n) is at most

$$n^c \frac{n!}{2^n}.$$

(Hint: Use Brégman's Theorem.)

Exercise 4 Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.