Discrete Mathematics III — So 2016

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Exercise Sheet 4

Due date: 12:30, May 25th, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Recall the standard construction from the lecture for the smallest unsatisfiable k-CNF formula on k variables: it used all 2^k clauses containing the same k-set of variables. How many variables are needed if we forbid that a k-set of variables is used for two different clauses? For $k > 0$ denote by $u(k)$ the least positive integer n, for which there exists an unsatisfiable k-CNF formula with n variables such that every k-set of variables appears in at most one clause. Prove that

$$u(k) = \frac{1}{\alpha} k(1 + o(1)),$$

where α is the solution of the equation $x = -x \log_2 x - (1-x)\log_2(1-x)$. ($H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. Hint: use that for large n and k we have that $\log_2 \binom{n}{k} \approx nH(\frac{k}{n})$.)

Exercise 2 Let X be a random variable taking integral non-negative values, let $\mathbb{E}[X^2]$ denote the expectation of its square, and let $\mathrm{Var}[X]$ denote its variance. Prove that

$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$

Exercise 3 Let X be a collection of pairwise orthogonal unit vectors in $\mathbb{R}^n$ and suppose the projection of each of these vectors on the first k coordinates is of Euclidean norm at least $\varepsilon > 0$. Show that $|X| \leq \dfrac{k}{\varepsilon^2}$, and this is tight for all $\varepsilon^2 = \dfrac{k}{2^r} < 1$.

Exercise 4 Recall that in the lecture we have seen, that the domination number of a graph on n vertices, with minimum degree δ is at most

$$n\frac{1 + \ln(\delta + 1)}{\delta + 1}.$$

a) Prove that if G is a d-regular graph with n vertices, then its domination number is at least

$$\frac{n}{d+1}.$$

b) Let $p = p(n)$ be such that $p \to 0$ and $d = np \to \infty$ as $n \to \infty$. Prove that with probability tending to 1 the domination number of the random graph $G(n, p)$ is at least

$$r = n\frac{\ln d}{d}(1 + o(1)).$$

Hint: First try to prove the following identities:

$$r = \log_q \left(\frac{d}{\ln^2 d}(1 + o(1)) \right), \text{ where } q = \frac{1}{1-p}$$

$$(1-p)^r = \frac{\ln^2 d}{d}(1 + o(1))$$

Exercise 5 Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta > 10$. Prove that there is a partition of V into two disjoint subsets A and B so that $|A| \leq O\left(\frac{n \ln \delta}{\delta}\right)$, and each vertex of B has at least one neighbour in A and at least one neighbour in B.