Discrete Mathematics III
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So 2016

Exercise Sheet 5

Due date: 12:30, June 8th, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Vocabulary: Let $E = (E_n)_{n\in\mathbb{N}}$ be a sequence of events in a probability space. We say that E holds *asymptotically almost surely (a.a.s.)* if $\mathbb{P}[E_n] \to 1$ as $n \to \infty$.

A graph property $\mathcal{P}$ is called *monotone* if $G \in \mathcal{P}$ and $G \subseteq F$ imply $F \in \mathcal{P}$. The function $p_0(n)$ is called a *threshold* for the monotone graph property $\mathcal{P}$, if

$$\mathbb{P}[G(n,p(n)) \text{ has property } \mathcal{P}) \to \begin{cases} 0 & \text{if } p(n) \ll p_0(n), \\ 1 & \text{if } p(n) \gg p_0(n). \end{cases}$$

An even faster change of behaviour with respect to the property $\mathcal{P}$ is formalized in the next definition.

A function $p_0(n)$ is called a *sharp threshold* of a monotone property $\mathcal{P}$ if for every $\epsilon > 0$

$$\mathbb{P}[G(n,p(n)) \text{ has property } \mathcal{P}) \to \begin{cases} 0 & \text{if } p(n) < (1-\epsilon)p_0(n), \\ 1 & \text{if } p(n) > (1+\epsilon)p_0(n). \end{cases}$$

For brevity, the parameter n of the function p is often omitted. A statement like "$G(n,p)$ is connected a.a.s.", just means that the probability that the random graph $G(n,p(n))$ is connected tends to 1.

Exercise 1 Prove that for every $\epsilon > 0$,

(a) $G(n,p)$ has no isolated vertices a.a.s., provided $p \geq (1+\epsilon)\frac{\ln n}{n}$.

(b) $G(n,p)$ is connected a.a.s., provided $p \geq (1+\epsilon)\frac{\ln n}{n}$.
(*Hint:* .yltnereffid meht eldnah dna ,n toorerauqs ,yas ,ta mus eht etarapes ;stuc fo rebmun detcepxeeht etamitsE)

Exercise 2 Prove that for every $\epsilon > 0$ the random graph $G(n,p)$ has an isolated vertex provided $p = (1-\epsilon)\frac{\ln n}{n}$. (*Hint:* .ytilauqeni s'vehsybehc esU)
Conclude that the properties $\mathcal{C}$ and $\mathcal{D}_1$ (being connected, having minimum degree at least 1) have the *same* sharp threshold function $\frac{\ln n}{n}$.

Exercise 3 Recall the proof of $R(3,k) = \Omega\left(\frac{k^2}{\log^2 k}\right)$ given in the lecture. Give a somewhat simpler/different argument for the second part of the argument: for a given vertex set $K \subseteq V$, to upper bound the probability of the existence of ℓ edge disjoint triangles, all of them sharing an edge with K, use the union bound for the expected number of such families of ℓ triangles and then Markov's inequality. This replaces the use of the Erdős-Tetali Lemma.

Exercise 4 Adapt the method of the lecture (or Exercise 3) to prove that the Ramsey number $R(4,k)$ satisfies

$$R(4,k) = \Omega\left(\left(\frac{k}{\ln k}\right)^{5/2}\right)$$

Remark. For comparison, the classic upper bound of Erdős and Szekeres gives $R(4,k) \leq \binom{k+4-2}{4-1} = O(k^3)$.