Discrete Mathematics III So 2016

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Exercise Sheet 6

Due date: 12:30, June 22nd, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 For $2 \le l < k < n$ denote by $M(n,k,l)$ the minimum size of a family of k-sets having the property that every l-set is contained in at least one of the k-sets. Similarly let $m(n,k,l)$ be the maximum size of a family of k-sets having the property that every l-set is contained in at most one of the k-sets. Prove that for fixed k, l

$$\lim_{n\to\infty} \frac{M(n,k,l)}{\binom{n}{l}/\binom{k}{l}} = 1 \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{m(n,k,l)}{\binom{n}{l}/\binom{k}{l}} = 1.$$

Exercise 2 Let X be a random variable with expectation $\mathbb{E}[X] = 0$ and variance σ^2. Prove that for all $\lambda > 0$,

$$\mathbb{P}[X \ge \lambda] \le \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

Exercise 3 Let $v_1 = (x_1, y_1), \dots, v_n = (x_n, y_n)$ be n two-dimensional vectors, where each x_i and y_i is an integer whose absolute value does not exceed $\dfrac{2^{n/2}}{100\sqrt{n}}$. Show that there are two non-empty disjoint sets $I, J \subset \{1, 2, \dots, n\}$ such that

$$\sum_{i\in I} v_i = \sum_{j \in J} v_j.$$

Exercise 4 Show that there is a positive constant c such that the following holds. For any n reals $a_1, \dots, a_n$ satisfying $\sum_{i=1}^{n} a_i^2 = 1$, if $(\varepsilon_1, \dots, \varepsilon_n)$ is a $\{-1, 1\}^n$-random vector obtained by choosing each ε_i randomly and independently with unifrom distribution to be either 1 or -1, then

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| \le 1\right] \ge c.$$