Discrete Mathematics III
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Exercise Sheet 7

Due date: 12:30, July 6th, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

For positive integers k and s a (k,s)-CNF formula is a k-CNF formula in which every variable appears in at most s clauses. For fixed k denote by $f(k)$ the largest integer s for which every (k,s)-CNF formula is satisfiable.

Exercise 1 Prove that $f(k) \geq k$ for all k.

Exercise 2 Improve the lower bound to an exponential function and show that

$$f(k) \geq \left\lfloor \frac{2^k}{ke} \right\rfloor - 1$$

for all k.

Exercise 3 Improve a factor two on the asymptotics and prove that

$$f(k) \geq \left\lfloor \frac{2^{k+1}}{(k+1)e} \right\rfloor$$

for all k.

Exercise 4 Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbours u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

Exercise 5 Prove that for every $\varepsilon > 0$ there is a finite $l_0 = l_0(\varepsilon)$ and an infinite sequence of bits $a_1, a_2, a_3, \ldots$, where $a_i \in \{0, 1\}$, such that for every $l > l_0$ and every $i \geq 1$ the two binary vectors $u = (a_i, a_{i+1,\ldots,a_{i+l-1}})$ and $v = (a_{i+l}, a_{i+l+1}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \varepsilon)l$ coordinates.

For hints, turn the page.

Hints

1. oN ytilibaborp si dedeen.

3. nI a (k, s)-CNF alumrof tes a elbairav v eurt htiw ytilibaborp $p_v = \frac{1}{2} + \frac{2d_{\overline{v}}-s}{2sk}$, erehw d_ℓ stneserper eht rebmun fo secnerrucco fo a laretil ℓ. nehT eht detagen elbairav $\overline{v}$ si deifsitas htiw ytilibaborp $p_{\overline{v}} = \frac{1}{2} - \frac{2d_{\overline{v}}-s}{2sk} \geq \frac{1}{2} + \frac{2d_v-s}{2sk}$, esuaceb $d_v + d_{\overline{v}} \leq s$. etoN eht evitiutniretnuoc erutan fo siht eciohc: eht erom a elbairav v srucco detagen, eht ssel ylekil ew lliw yfsitas $\overline{v}$.