Discrete Mathematics III — So 2016
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Exercise Sheet 8

Due date: 12:30, July 20th, at the beginning of the exercise class.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$, independently, is connected (and spanning). Let Q denote the probability that in a random two-coloring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

Exercise 2 A family of subsets of $\mathcal{G}$ is called intersecting if $G_1 \cap G_2 \neq \emptyset$ for all distinct $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be k intersecting families of subsets of $\{1, 2, \ldots, n\}$. Prove that

$$\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| \leq 2^n - 2^{n-k}.$$

Exercise 3 Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k-1$ is at least $1/4^k$.

Exercise 4 Let $A_1, A_2, \ldots, A_k$ be fixed subsets of $[n]$ and $p_1, p_2, \ldots, p_n$ real numbers from $[0, 1]$. In the lecture we have seen that if R is the random subset of $[n]$ that we get by putting the element i into R with probability p_i, independently for all elements, then

$$\mathbb{P}(R \cap A_i \neq \emptyset \text{ for every i}) \geq \prod_{i=1}^{k} \mathbb{P}(R \cap A_i \neq \emptyset). \tag{1}$$

Now let R be an ℓ-element subset of $[n]$ chosen uniformly at random from all ℓ-element subsets (for some $1 \leq \ell \leq n$). Give examples of sets $A_1, A_2, \ldots, A_k$ where

a) (1) does not hold anymore.

b) (1) still holds.

Exercise 5 State and prove the FKG inequality for

a) two decreasing properties.

b) one decreasing and one increasing property.