Discrete Mathematics III
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So 2016

Bonus Exercise Sheet

Due date: at the beginning of the exam.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Let $G = (V, E)$ be the graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent iff they differ in precisely one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.

Exercise 2 Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly at random from all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that

$$\mathbb{P}(\chi(H) \leq 400) < 1/100.$$

Exercise 3 Prove that there is an absolute constant c such that or every $n > 1$ there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the probability that the chromatic number of $G(n, 1/2)$ lies in I_n is at least 0.99.