The Lemma. *For each* $x \in \mathbb{C}_+^n : |p(x)| \geq |p(\mathrm{Re}(x))|$.

The Remark. *For each* $y \in \mathbb{C}_{++}^{n-1}, \prod_{i=1}^{n-1} \mathrm{Re}(y_i) = 1, t \in \mathbb{R}_{++} : \mathrm{cap}(p) \leq \frac{p(\mathrm{Re}(y),t)}{t}$.

The Theorem. *Let* $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ *be H-stable and homogeneous of degree* n*. Then* $p' \equiv 0$ *or* p' *is H-stable and homogeneous of degree* $n-1$*. Furthermore,*

$$\mathrm{cap}(p') \geq \mathrm{cap}(p) g(k), \text{ where } k := \deg_{x_n}(p).$$

Three things to show

1. p' is homogeneous of degree $n-1$ if $p' \not\equiv 0$✓
2. If $p' \not\equiv 0$, then p' H-stable
3. $\mathrm{cap}(p') \geq \mathrm{cap}(p) g(k)$

- For H-stability, we need to show that p' has no roots in $\mathbb{C}_{++}$.
  - Equivalently, as soon as some $y \in \mathbb{C}_{++}$ is a root of p', then $p' \equiv 0$.
  - It suffices to show this for those y with $\prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1$ only:
  - As p' is homogeneous, $p'(y) = 0 \iff p'(\lambda y) = \lambda^{n-1} p'(y) = 0, \lambda \in \mathbb{R}_{++}$
- For the capacity, let's look at the definition again:
  - $\mathrm{cap}(p') := \inf\{p'(y) : y \in \mathbb{R}_{++}^{n-1}, \prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1\}$
  - If we succeed in showing $p'(y) \geq \mathrm{cap}(p) g(k)$ holds for all $y \in \mathbb{R}_{++}^{n-1}, \prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1$, we may deduce the desired statement directly.

Therefore, we will go through all $y \in \mathbb{C}_{++}^{n-1}, \prod_{j=1}^{n} \mathrm{Re}(y_j) = 1$ and show these two properties:

- (Property A) If y is a root of p', then $p' \equiv 0$
- (Property B) If $y \in \mathbb{R}_{++}^{n-1}$, then $p'(y) \geq \mathrm{cap}(p) g(k)$, where $k = \deg_{x_n} p(x)$.

We can then look at the values of $p(y, 0), p(y, t), t > 0$

- categorise these y in the following way:

	$\deg_t p(y,t)$ ≤ 1	$\deg_t p(y,t)$ ≥ 2
$p(y,0) = 0$	Case 1&2	Case 1
$p(y,0) \neq 0$	Case 2	Case 3

1 Case 1: $p(y,0)=0$

We saw in our Lemma that $\forall x \in \mathbb{C}_+^n : |p(x)| \geq |p(\mathrm{Re}(x))|$. This has two applications here:

$$0 \leq |p(\mathrm{Re}(y),0)| \overset{\text{Lemma}}{\leq} |p(y,0)| \overset{\text{Case 1}}{=} 0 \Rightarrow p(\mathrm{Re}(y),0) = 0 \tag{1}$$

$$p(\mathrm{Re}(y),t) \leq |p(\mathrm{Re}(y),t)| \overset{\text{Lemma}}{\leq} |p(y,t)| \tag{2}$$

We can say the following about p':

$$p'(y) \overset{\text{Def } p'}{=} \lim_{t\searrow 0} \frac{p(y,t)-p(y,0)}{t} \overset{\text{Case 1}}{=} \lim_{t\searrow 0} \frac{p(y,t)}{t} \tag{3}$$

$$p'(\mathrm{Re}(y)) \overset{\text{Def } p'}{=} \lim_{t\searrow 0} \frac{p(\mathrm{Re}(y),t)-p(\mathrm{Re}(y),0)}{t} \overset{(1)}{=} \lim_{t\searrow 0} \frac{p(\mathrm{Re}(y),t)}{t} \tag{4}$$

Now we can use what we know and put it all together:

$$\left.\begin{array}{r} p'(\mathrm{Re}(y)) \overset{(4)}{=} \\ \\ \\ \mathrm{cap}(p) \overset{\text{Remark}}{\leq} \end{array}\right\} \lim_{t\searrow 0} \frac{p(\mathrm{Re}(y),t)}{t} \overset{(2)}{\leq} \lim_{t\searrow 0} \frac{|p(y,t)|}{t} \overset{(3)}{=} |p'(y)|$$

- To prove Property A:

  $$p'(y) = 0 \Rightarrow |p'(y)| = 0 \Rightarrow p'(\mathrm{Re}(y)) \leq 0$$

  - As all the coefficients of p are non-negative, so are all the coefficients of p'.
  - But as $y \in \mathbb{C}_{++}$, this is only possible if $p' \equiv 0$.

- To prove Property B: Assume $y \in \mathbb{R}_{++}^{n-1}$, then

$$\begin{aligned} p'(y) &= |p'(y)| && \text{because } p \text{ only has real positive coefficients} \\ &\geq \mathrm{cap}(p) && \text{due to our result above} \\ &\geq \mathrm{cap}(p)g(k) && \text{as } 0 \leq g(k) \leq 1 \end{aligned}$$

2 Case 2: $\deg_t p(y,t) \leq 1$

Although the criterion for this case is very different, it works very similarly.

- By our Lemma, $p(\mathrm{Re}(y), t) \leq |p(y,t)|$.
- This implies that $p(\mathrm{Re}(y), t)$ has degree at most 1 in t as well.
- Recall that p' retains exactly the terms that are linear in x_n

$$p'(y) = \lim_{t\to\infty} \frac{p(y,t)}{t} \tag{5}$$

$$p'(\mathrm{Re}(y)) = \lim_{t\to\infty} \frac{p(\mathrm{Re}(y),t)}{t} \tag{6}$$

- Now we can put it together in the exact same way:

$$\left.\begin{array}{r} p'(\mathrm{Re}(y)) \overset{(6)}{=} \\ \\ \mathrm{cap}(p) \overset{\text{Remark}}{\leq} \end{array}\right\} \boxed{\lim_{t\to\infty}} \frac{p(\mathrm{Re}(y),t)}{t} \overset{\text{Lemma}}{\leq} \boxed{\lim_{t\to\infty}} \frac{|p(y,t)|}{t} \overset{(5)}{=} |p'(y)|$$

- This proves property A and B exactly as in Case 1.

3 Case 3: $p(y,0) \neq 0, \deg_t p(y,t) \geq 2$

- Fixing the y cannot increase the degree of t, so $k := \deg_{x_n} p(x) \geq 2$
- We can rewrite our homogeneous polynomial $p(y,t)$ using some $a_1, \ldots, a_k \in \mathbb{C}$:

$$p(y,t) = p(y,0) \prod_{i=1}^{k} (1 + a_i t) \tag{7}$$

- We may read off the linear terms of the derivative, determine $p'(y)$ directly:

$$p'(y) = p(y,0) \sum_{i=1}^{k} a_i \tag{8}$$

- As we know the degree of t is at least 2, not all a_i can be zero.

The Claim. *If $a_i \neq 0$, then a_i^{-1} is a non-negative linear real combination of $y_1, \ldots, y_{n-1}$.*

We will explain this claim later, look at its consequences first:

- $y \in \mathbb{C}_{++}$, so $\mathrm{Re}(y_j) > 0$
- Then the Claim implies: If $a_i \neq 0$, then $\mathrm{Re}(a_i) > 0$.
- We know that not all a_i are zero $\Rightarrow \sum_{i=1}^{k} a_i \neq 0$ because its real part is strictly positive.
- $p(y,0)$ is nonzero by the case we are in
- Therefore, $p'(y) = p(y,0) \sum_{i=1}^{k} a_i \neq 0$ and we showed Property A ✓

The second thing to show is that if $y \in \mathbb{R}_{++}^{n-1}, \prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1$ then $p'(y) \geq \mathrm{cap}(p) g(k)$.

- By the claim, if all entries of y are positive reals, all nonzero a_i are positive reals.

$$\Rightarrow \sum_{i=1}^{k} a_i \in \mathbb{R}_{++} \Rightarrow \frac{p(y,0)}{p'(y)} = \frac{1}{\sum_{i=1}^{k} a_i} \in \mathbb{R}_{++}$$

- This allows us to cleverly set a positive real t:

$$t := \frac{k}{k-1} \frac{p(y,0)}{p'(y)} \in \mathbb{R}_{++} \tag{9}$$

We now play around with the arithmetic-geometric mean inequality to get a helpful information:

$$\begin{aligned}
\frac{p(y,t)}{p(y,0)} &= \prod_{i=1}^{k}(1+a_i t) && \text{by (7)} \\
&\leq \left(\frac{1}{k}\sum_{i=1}^{k}(1+a_i t)\right)^k && \text{by GM-AM inequality} \\
&= \left(\frac{1}{k}(k+t\frac{p'(y)}{p(y,0)})\right)^k && \text{by (8)} \\
&= \left(1+\frac{t}{k}\frac{p'(y)}{p(y,0)}\right)^k \\
&= \left(1+\frac{1}{k-1}\right)^k && \text{by Def. t} \\
&= (\frac{k}{k-1})^k
\end{aligned}$$

We may use this inequality and apply our Remark:

$$\begin{aligned}
\operatorname{cap}(p) &\leq \frac{p(\operatorname{Re}(y),t)}{t} && \text{by Remark} \\
&= \frac{p(y,t)}{t} && \text{as } y \in \mathbb{R}^{n-1}_{++} \\
&= p'(y)\frac{k-1}{k}\frac{p(y,t)}{p(y,0)} && \text{by Definition of t} \\
&\leq p'(y)\frac{k-1}{k}(\frac{k}{k-1})^k && \text{by the above calculation} \\
&= p'(y)(\frac{k}{k-1})^{k-1} \\
&= \frac{p'(y)}{g(k)} && \text{by Definition of } g(k)
\end{aligned}$$

This gives us precisely the inequality $p'(y) \geq \operatorname{cap}(p)g(k)$ we were looking for.
We are done with the proof, if our claim about the a_i holds. Let's prove that claim:

The Claim. *If $a_i \neq 0$, then a_i^{-1} is a non-negative linear real combination of $y_1, \ldots, y_{n-1}$.*

Proof. We use the Farkas lemma.

- Maybe you remember it from a course on Discrete Geometry or Linear Programming
- Comes in many different shapes and forms to achieve different goals
  - determine whether a polyhedron lies in a certain halfspace
  - find a hyperplane separating a point from a polyhedron
  - less geometrical: determine whether a set of inequalities has a non-negative solution.

Farkas Lemma. *Let $A \in \mathbb{R}^{r \times s}, b \in \mathbb{R}^r$. Exactly one of the following holds:*

1. *$Ax = b, x \geq 0$ has a solution.*
2. *$\bar{x}A \geq 0, \bar{x}b < 0$ has a solution.*

- We fix $r = 2, s = n-1$, an arbitrary $1 \leq i \leq n-1$ such that $a_i \neq 0$ and:

$$A = \begin{pmatrix} \mathrm{Re}(y_1), & \ldots, & \mathrm{Re}(y_{n-1}) \\ \mathrm{Im}(y_1), & \ldots, & \mathrm{Im}(y_{n-1}) \end{pmatrix} \quad b = \begin{pmatrix} \mathrm{Re}(a_i^{-1}) \\ \mathrm{Im}(a_i^{-1}) \end{pmatrix}$$

- Then the first alternative of the Farkas lemma states that the vector b may be displayed as a non-negative real linear combination of the columns of the matrix A. This is exactly what we claim: If a_i is nonzero, then its inverse is a non-negative real linear combination of the $y_1, \ldots, y_{n-1}$.
- Therefore, disproving the second alternative proves the claim. We assume the second alternative holds for some $\bar{x} = (c, d)$. If we can lead this to a contradiction, we are done.
- We may assume that even $\bar{x}A > 0$ holds strictly, otherwise we could just add a small ε to c as all $\mathrm{Re}(y_j)$ are strictly positive.
- We want a contradiction to the H-stability of p: Find $z \in \mathbb{C}_{++}^n$ that is a root of p.
- We introduce $\lambda := c - i \cdot d$ and look at the vector $z := \lambda(y, -a_i^{-1})$. Then $z \in \mathbb{C}_{++}^n$:
  - $\mathrm{Re}(\lambda y_j) = c\,\mathrm{Re}(y_j) + d\,\mathrm{Im}(y_j)$ is the j-th entry of $\bar{x}A > 0$.
  - $\mathrm{Re}(\lambda(-a_i^{-1})) = -\mathrm{Re}(\lambda a_i^{-1}) = -c\,\mathrm{Re}(a_i^{-1}) - d\,\mathrm{Im}(a_i^{-1}) = -\bar{x}b > 0$.
- On the other hand, as p is homogeneous, $p(z) = 0$:

$$p(z) = \lambda^n p(y, -a_i^{-1}) = \lambda^n p(y, 0) \prod_{j=1}^{k} (1 + a_j(-a_i^{-1})) = 0 \tag{10}$$

  where in the last step, for $j = i$, the product collapses.

□