Designs and Codes | So 2017
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Exercise Sheet 1

Due date: 12:30, May 3rd, at the beginning of the exercise class.
Late submissions will be used to soundproof my office.

You should try to solve all of the exercises below, and submit **three** solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 This exercise provides some further examples of designs.

(a) Let the ground set be the edges of K_6. Take the blocks to be all copies of K_3 and all matchings of size three. Show that this gives a 2-design, and determine its parameters v, k and λ.

(b) Let the ground set be the edges of K_7. Suppose we take as blocks all copies of $K_{1,5}$, all copies of C_5, and all copies of some graph F. What does F need to be to give a 3-design, and what are the other parameters?

Exercise 2 For $m \geq 3$, let $X = \mathbb{F}_2^m \setminus \{\vec{0}\}$, and take $\{\vec{x}, \vec{y}, \vec{z}\}$ to be a block if and only if $\vec{x} + \vec{y} + \vec{z} = \vec{0}$.

(a) Show that this gives a Steiner triple system; that is, a 2-$(2^m - 1, 3, 1)$ design.

(b) If $m = 3$, show that this design is isomorphic to the Fano plane.

Exercise 3 Prove Proposition 2: if a non-trivial t-$(v, k, 1)$ design exists, then we must have $v \geq (t+1)(k-t+1)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S1.html`.]

Exercise 4 Complete the proof of Proposition 5 by showing that a projective plane $(P, \mathcal{L})$ of order[1] n gives a symmetric 2-$(n^2 + n + 1, n + 1, 1)$ design with $X = P$ and $\mathcal{D} = \mathcal{L}$.

[1] The *order* of a projective plane is defined as the number of points in a line minus one. As you shall have to prove, any two lines in a projective plane contain the same number of points, so it does not matter which line you choose.

Exercise 5 In this exercise we shall consider 2-($\binom{k+1}{2}, k, 2$) designs.

(a) Find an example with $k = 3$. It may help to place five of the points on a circle.

(b) Let $B_1, B_2, \ldots, B_b$ be the blocks of the design, and for $2 \leq i \leq b$, let $\mu_i = |B_1 \cap B_i|$. Compute

  (i) $\sum_{i=2}^{b} \mu_i$,

  (ii) $\sum_{i=2}^{b} \mu_i(\mu_i - 1)$, and

  (iii) $\sum_{i=2}^{b} (\mu_i - 1)(\mu_i - 2)$

  in terms of k. What can you say about the μ_i's?

Exercise 6 In lecture we noted that taking λ copies of a t-$(v, k, 1)$ design gives a $t-(v, k, \lambda)$ design. However, this resulting design contains repeated blocks and is thus not simple. Show that if $2t \leq k < v$, the existence of a t-$(v, k, 1)$ design implies the existence of a *simple* t-$(v, k, 2)$ design.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S1.html`.]