Designs and Codes — So 2017
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Exercise Sheet 2

Due date: 12:30, May 17th, at the beginning of the exercise class.
Late submissions will not earn any credit.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Let $\mathcal{D}$ be a 2-$(n^2, n, 1)$ design for some $n \in \mathbb{N}$. Prove that the blocks of $\mathcal{D}$ can be partitioned into $n+1$ parts, each part containing n blocks, such that:

(i) Any two blocks from the same part are disjoint.

(ii) Two blocks from different parts have exactly one element in common.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S2.html`.]

Exercise 2 Let $\mathcal{D}$ be a t-(v, k, λ) design, and suppose we have $i, j \geq 0$ with $i + j \leq t$.

(a) Show that for any i-set $I \in \binom{X}{i}$ and j-set $J \in \binom{X}{j}$ with $I \cap J = \emptyset$, there are exactly $b_{i,j}$ blocks $B \in \mathcal{D}$ with $I \subseteq B$ and $B \cap J = \emptyset$, where

$$b_{i,j} = \frac{\lambda \binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}.$$

(b) Prove that for any $0 \leq a \leq t$, we have

$$\lambda_{t-a} = \sum_{i=0}^{a} \binom{a}{i} b_{t-i,i},$$

where λ_{t-a} is the number of blocks containing a fixed $(t-a)$-set.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S2.html`.]

Exercise 3 Recall that $W_{t,k}$ is the higher incidence matrix N_t for the trivial design $\mathcal{D} = \binom{X}{k}$, so its rows are indexed by all $\binom{v}{t}$ t-sets, and its columns by all $\binom{v}{k}$ k-sets.

(a) Prove that if $v \leq k + t$, then the columns of $W_{t,k}$ are linearly independent.

(b) Use part (a) to give an alternative proof of the fact that any t-(v, k, λ) design with $v \leq t + k$ must be trivial.

Exercise 4 Let $(G,+)$ be a commutative group with $|G| = v$. A family $\mathcal{B}$ of k-subsets $\{B_i : i \in I\}$, where for each $i \in I$, $B_i = \{g_{i,1}, g_{i,2}, \dots, g_{i,k}\} \subseteq G$, is called a *difference family* with parameters (v,k,λ) if every non-zero element of G appears exactly λ times in the multiset of differences

$$\Delta\mathcal{B} = \{g_{i,j} - g_{i,j'} : i \in I; j \neq j' \in [k]\}.$$

(a) Show that for $G = \mathbb{Z}_9$, $B_1 = \{0,1,2,5\}$ and $B_2 = \{0,2,3,5\}$ form a difference family. What are its parameters?

Given a family of k-subsets $\mathcal{B} = \{B_i : i \in I\}$, define $\mathcal{D}(\mathcal{B}) = \{g + B_i : g \in G, i \in I\}$ (take this to be a multiset, if needed, so that $|\mathcal{D}(\mathcal{B})| = |G|\,|I|$), where for any set S, we have $g + S = \{g + s : s \in S\}$.

(b) Prove that $\mathcal{B}$ is a difference family in G with parameters (v,k,λ) if and only if $\mathcal{D}(\mathcal{B})$ is a 2-(v,k,λ) design.

Exercise 5 In class we saw how Hadamard matrices correspond to Hadamard designs, which are symmetric 2-(v,k,λ) designs with $v = 4(k-\lambda)-1$. Show that, given any normalised $4n \times 4n$ Hadamard matrix H, one can build a *Hadamard* 3*-design*, which is a 3-$(4n, 2n, n-1)$ design.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S2.html`.]

Exercise 6 Hadamard proved that any $n \times n$ matrix A whose entries satisfy $a_{i,j} = \pm 1$ has $|\det A| \leq n^{n/2}$. Show that $|\det A| = n^{n/2}$ if and only if A is a Hadamard matrix.