Exercise Sheet 3

Due date: 12:30, May 31st, at the beginning of the exercise class.
Late submissions will fare worse than San Marino at Eurovision 2017.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1

(a) Let $(G, \cdot)$ be a finite commutative group with $|G|$ odd. Define $(G, \square)$ by $x \square y = x \cdot y^{-1}$. Show that the Latin squares made up by the tables of $(G, \cdot)$ and $(G, \square)$ are orthogonal.[1]

(b) Let $L : \mathbb{Z}_{2n} \times \mathbb{Z}_{2n} \to \mathbb{Z}_{2n}$ be given by $L(i, j) = i + j \pmod{2n}$. Show that L is a Latin square without an orthogonal mate. Even more, show that L does not have a single transversal.

Exercise 2 A *magic square* Q of order n is an $n \times n$ square containing the numbers $1, \ldots, n^2$ in such a way that all row sums, column sums and diagonal sums are the same. Q is *semi-magic* if the condition on the diagonals is dropped. How can a pair of orthogonal Latin squares on $\{1, \ldots, n\}$ be used to construct a semi-magic square?

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S3.html`.]

Exercise 3 Consider a $TD(t, n)$ on $(X, \mathcal{G} = \{G_1, \ldots, G_t\}, \mathcal{B})$. Prove:

(a) Every element is in exactly n blocks of $\mathcal{B}$.

(b) $|\mathcal{B}| = n^2$.

(c) If $t = n + 1$, then any two blocks intersect.

(d) If $t < n + 1$, then every block is disjoint from some other block.

[1] The rows and columns of the two tables should be indexed in the same way.

Exercise 4 A Latin square L on $\{1, \ldots, n\}$ is *idempotent* if $L(i,i) = i$ for all i. Show that idempotent Latin squares of all orders $n \neq 2$ can be constructed as follows:

(a) Let $n \geq 3$ be odd. Define L by $L(i,j) = 2i - j \pmod{n}$.

(b) Let $n \geq 4$ be even. Take L of order $n-1$ as in (a), and append an nth row and column appropriately.

Exercise 5 A Latin square L is *self-orthogonal* if L and L^T are orthogonal. Let $n = p^r$ be a prime power, $n \neq 2, 3$, and write $GF(n) = \{a_0, a_1, \ldots, a_{n-1}\}$,[2] where $a_0 = 0$. Let $\alpha \in GF(n) \setminus \{0, 1\}$ be such that $2\alpha \neq 1$. Define L by $L(i,j) = \alpha a_i + (1 - \alpha) a_j$, and show that L is a self-orthogonal Latin square.

Exercise 6 Check that the following construction yields a self-orthogonal Latin square of order 14 (thus proving $N(14) \geq 2$). Take the row R (of length 13) below and develop it cyclically modulo 13, where $\infty + 1 = \infty$.

$$
\begin{array}{cccccccccccccc}
R: & 0 & 8 & 3 & 12 & 9 & 2 & 5 & 10 & 6 & 11 & 1 & 4 & \infty \\
 & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 & \infty & 1 & 9 & 4 & 0 & 10 & 3 & 6 & 11 & 7 & 12 & 2 & 5 \\
 & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 & 6 & \infty & 2 & 10 & 5 & 1 & 11 & 4 & 7 & 12 & 8 & 0 & 3 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots &
\end{array}
$$

Repeat this until we obtain a 13×13 matrix. Append to this a fourteeth row and column to form a Latin square on $\{0, 1, \ldots, 12, \infty\}$, and show that L is self-orthogonal.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2017/hints/S3.html`.]

[2] $GF(n) = \mathbb{F}_n$, the finite field of order n.