Designs and Codes | So 2017
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Exercise Sheet 4

Due date: 12:30, Jun 14th, at the beginning of the exercise class.
If you submit your homework late, you're gonna have a bad time.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 A Hadamard matrix is *regular* if all row and column sums are the same. We shall construct a regular symmetric $4t^2 \times 4t^2$ Hadamard matrix as follows. Let M be an $OA(t, 2t)$, and define $H = (h_{i,j})$ by $h_{i,i} = 1$, and

$$h_{i,j} = \begin{cases} -1 & \text{if the } i\text{th and } j\text{th column of } M \text{ agree in some row} \\ 1 & \text{otherwise.} \end{cases}$$

Prove:

(a) H is regular with $2t^2 + t$ positive entries in each row.

(b) For any two rows of H, there are $t^2 - t$ columns in which they both have negative entries.

(c) Any two rows of H differ in exactly $2t^2$ entries

Deduce from this that H is a regular, symmetric Hadamard matrix.

Exercise 2 Complete the proof of Theorem 3; that is, show that the final array constructed is indeed an $OA(4, 3m + 1)$ on the symbols $\{0, 1, \ldots, 2m\} \cup \{x_1, \ldots, x_m\}$.

Exercise 3 Complete the proof of Theorem 4 by showing that the final array is truly an $OA(k, mt + u)$.

Exercise 4 Similar to Theorem 4, it is also true that for $0 \leq u, v \leq t$,

$$N(tm + u + v) \geq \min \left(N(t) - 2, N(m), N(m + 1), N(m + 2), N(u), N(v) \right),$$

where we take $N(0) = \infty$.

Suppose we have some M such that $N(n) \geq 3$ for all $M \leq n \leq 7M + 3$. Show that for all $7M \leq n \leq 7^2M + 3$ we also have $N(n) \geq 3$, and deduce that $N(n) \geq 3$ for all $n \geq M$.

Exercise 5 Suppose $N(12) \geq 3$ and $N(24) \geq 3$. Use Corollary 3 to deduce that we then have $N(4m) \geq 3$ for all $m \geq 1$.

Exercise 6 Two tennis clubs, each with n players, wish to play a competition in which doubles are played, with two players from one club opposing two from the other club, subject to the following conditions:

(i) Every player must play against every member of the other club exactly once.

(ii) Two members of the same club can be partners in at most one game.

Show that $n = 2m$ must be even, and that a match schedule exists provided that $N(m) \geq 2$. What about $m = 2$ or 6?