Constructive Combinatorics So 2018
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Bonus Exercise Sheet

Due date: July 30th

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Establish that the construction we gave using the Abbott-product of graphs from the d-wise independent sample space we described is indeed strongly explicit. That is, for any n define a graph G_n that is $2^{\frac{\log\log\log n}{\log\log n}}$-Ramsey, and describe and algorithm that outputs in polylog(n)-time whether two input vertices u and $v \in [n]$ are adjacent in G_n or not.

Exercise 2 The construction of Codenotti–Pudlak–Resta we saw gave a constructive lower bound of $R(3, k)$ of order $k^{4/3}$. In this exercise, we seek to improve this bound by making the following changes to the construction G:

- Instead of starting with the girth-8 Benson graph, take the underlying graph B to be the point/line incidence graph of the projective plane.
- As before, the edges of the underlying graph will be the vertices of our construction; that is, $V(G) = E(B)$.
- Fix an arbitrary ordering $\prec$ on the set P of points of the projective planes.
- Put an edge between vertices $p_1\ell_1$ and $p_2\ell_2$ of G if $p_1 \prec p_2$, $\ell_1 \neq \ell_2$, and $p_1\ell_2 \in E(B)$.

Show that this graph gives a constructive lower bound of order $k^{3/2}$ for $R(3, k)$.

Exercise 3 The bipartite Ramsey number $BR(k, k)$ denotes the smallest integer N such that every two-coloring of $K_{N,N}$ contains a monochromatic $K_{k,k}$. Similarly to the non-bipartite case, the random two-coloring shows that $BR(k, k) > \sqrt{2}^k$. The parallels stop right there though, as comparable constructive lower bounds are much harder to obtain. Abbott's product, Nagy's set intersection construction, and even the simple Turán's construction has no obvious analogue. Even for a construction of quadratic size, we have to work some.

A square matrix H with entries $+1$ and -1 is called an *Hadamard matrix* if the rows are pairwise orthogonal.

(a) Show that that the colums of an Hadamard matrix are pairwise orthogonal.

(b) Show that the order of an Hadamard matrix is 1 or 2 or divisible by 4.[1]

(c) Construct $2^n \times 2^n$ Hadamard matrices for every integer $n \geq 1$.

(d) Given an Hadamard matrix $H = (h_{ij})$, define a two-coloring of $K_{N,N}$ by coloring the edge xy red if the entry $h_{x,y}$ is $+1$ and blue otherwise. Prove that for arbitrary integers s, t, $1 \leq s, t \leq N$, we have

$$\left| \sum_{i=1}^{r} \sum_{j=1}^{s} h_{i,j} \right| \leq \sqrt{rsN}.$$

Conclude that this coloring is $\sqrt{N}$-Ramsey.

[Hint (to be read backwards): zrawhcS-yhcuaC esU]

Exercise 4 Let q be the power of an odd prime number and let $\rho_q : \mathbb{F}_q \to \{1, -1\}$ denote the quadratic chracter, extended to 0 by $\rho_q(0) = 1$. The rows and columns of the matrix $Q = (q_{a,b})$ are labeled by the elements of $\mathbb{F}_q$ and its entries are defined by $q_{a,b} = \rho_q(a - b)$. Let H be the $(q + 1) \times (q + 1)$-matrix we obtain by adding a row and column of 1s to Q. Prove that H is an Hadamard matrix.

[1] It is a notorious conjecture of design theory that Hadamard matrices exist for all N divisible by 4.