Chapter 3

Even Cycles

In this chapter we go on studying graphs free of some fixed even cycle. This choice seems natural as, besides complete bipartite graphs, even cycles are the other front-runner for the title of "arguably the simplest family of bipartite graphs". Note that $K_{2,2} = C_4$ is the member of both families. You'll find other similarities between complete bipartite graphs and even cycles: our knowledge of their Turán numbers are equally scarce.

Graphs without short cycles were studied much earlier in a different context, mostly in the context of finite geometry and group theory. More recently they popped up in computer science in connection with "expander graphs", i.e., graphs in which every "small" subset of the vertices has a "large" neighborhood (the set "expands"). While having no short cycle is not required of an expander, it just so happens that some of the best expanders tend to have large girth[1] and thus are relevant in our context as well. This is not a complete coincidence though: the ultimate d-regular expander, the infinite $d-1$-ary tree, has "infinite girth".

3.1 The Moore-bound

What would be a first idea to give an upper bound on the number of edges in a $2k$-cycle free graph G?

Here is a heuristic "proof" that $ex(n, C_{2k}) \lessapprox \frac{1}{2}n^{1+\frac{1}{k}}$. Let G be a C_{2k}-free graph with average degree $\bar{d}$. We fix a vertex v. Since there is no $2k$-cycle in G, any two disjoint paths of length k starting at v must have different other endpoints. (If there were two such paths with identical endpoints, their union would create a $2k$-cycle.) Now let's count paths of length k starting at v. If v is chosen properly, then it has roughly $\bar{d}$ neighbors, all its neighbors have roughly $\bar{d}-1$ further neighbors, etc... Hence there are roughly $(\bar{d}-1)^k$ paths starting at v, all of which must have a different endpoint. Thus $n \geq (\bar{d}-1)^k$, which implies $\bar{d} \leq n^{1/k}+1$, and we are done. Right?

Well, this was easy, except one needs to hammer out some of the details... There are two problems with the above argument, each of them is centred around the appearance of the word "roughly". First of all we need to find a vertex v, that has the property that most of the vertices which have distance at most $k-1$ to it have degree roughly

[1] The *girth* of a graph is the length of its shortest cycle.

$\bar{d}$. This could be overcome for the moment by assuming that G is d-regular, i.e. $\bar{d} = d$. The second problem in the proof arises when we claim that the $(\bar{d}-1)^k$ paths of length k starting at v all have distinct endpoints. Some of these paths may intersect each other *earlier*, which means that even if their endpoint were the same, their union wouldn't form a $2k$-cycle. Hence we cannot infer the crucial lower bound on the number of vertices. This difficulty will disappear the moment we assume not only that G contains no cycle of length $2k$, but also that it doesn't contain shorter cycles, i.e. that its girth is *at least* $2k+1$.

To conclude, the above heuristic argument does prove the following classical combinatorial result.

Proposition 3.1 *(Moore bound) Let G be a d-regular graph of girth $2k+1$. Then*

$$n(G) \geq \sum_{i=0}^{k-1} (d-1)^i d + 1.$$

Exercise 3.1 *Derive the Moore-bound for even girth. (We sometimes refer to this result as the* even-Moore bound*). That is, show that if G is a d-regular graph with girth $2k$, then*

$$n(G) \geq 2 \sum_{i=0}^{k-1} (d-1)^i.$$

In particular the Moore-bound would show that $ex(n, C_3, C_4, \ldots, C_{2k}) \leq \frac{1}{2} n^{1+1/k} + \frac{1}{2} n$ had we not had the convenient assumption of the regularity of G. Alon, Hoory, and Linial (2002) overcame this difficulty and showed that the Moore-bound is valid for irregular graphs as well. We will discuss their result in Section 3.4, for now we are content with a weaker version which makes the above heuristic precise.

For this purpose we introduce a simple trick to pass from average degree to minimum degree. This claim is very useful in general, in a wide variety of settings, where we only care about the order of magnitude of the final answer.

Claim 3 *Let G be a graph with average degree $\bar{d}(G) = d$. Then there is a subgraph $H \subseteq G$ such that $\delta(H) > d/2$.*

Proof. If we find a vertex v in G with degree $d(v) \leq d/2$, we kick it out of G. We repeat this process until there is no such vertex, and we denote the graph obtained by H. This H is certainly what we want, unless it is empty. How many edges did we delete in the process? In each step at most $d/2$ edges were kicked out, altogether at most $(n-1)d/2$. Since $e(G) = \frac{d}{2} n$, not all edges were deleted. Thus H is not empty and has no vertex of degree at most $d/2$. □

Proposition 3.2 $ex(n, C_3, \ldots, C_{2k}) \leq n^{1+1/k} + n$.

Proof. Let G have n vertices, average degree d and girth at least $2k+1$. By Claim 3, G has a subgraph $H \subseteq G$ with minimum degree $\delta(H) \geq d/2$. Obviously, the girth of H is also at least $2k+1$. Then by the above heuristic argument (which is now precise if we replace $\bar{d}$ with $\delta = \delta(H)$)

$$\begin{aligned} n \geq n(H) &\geq (\delta-1)^k, \\ n^{1/k}+1 &\geq \delta \geq d/2, \\ n^{1+1/k}+n &\geq nd/2 = e(G) \end{aligned}$$

and we are done. □

Exercise 3.2 *(i) Show that for any tree T on k vertices, the Turán number is linear in n. More precisely, for $n \geq k$ we have*

$$ex(n,T) \leq n(k-2).$$

(ii) Exhibit a tree S on k vertices which has Turán number $ex(n,S) = n(k-2)/2$ for infinitely many n.

Remark. *The Erdős-Sós conjecture (1962) states that $n(k-2)/2$ is equal to the Turán number of* any *tree on k vertices. Recently Ajtai, Komlós, Simonovits, and Szemerédi announced that they proved the conjecture.*

3.2 Upper bound for the size of C_{2k}-free graphs

Clearly, $ex(n, C_3, \ldots, C_{2k}) \leq ex(n, C_{2k})$. Bondy and Simonovits (1974) were the first to prove that $ex(n, C_{2k})$ is also of order at most $n^{1+\frac{1}{k}}$. The constant of the leading term depended on k (it was $90k$), not like in the simple upper bound of Proposition 3.2 for $ex(n, C_3, \ldots, C_{2k})$. Later Verstraëte polished the argument of Bondy and Simonovits and obtained the current best constant $8(k-1)$. This is the proof we include here.

Theorem 3.3 $ex(n, C_{2k}) \leq 8(k-1)n^{1+1/k}$.

Proof. By Theorem 2.1 we can assume that $k \geq 3$. Let G' be a graph with at least $8(k-1)n^{1+1/k}$ edges. We find a $2k$-cycle in G'.

First we take a bipartite subgraph $G \subseteq G'$ with $4(k-1)n^{1+1/k}$ edges.

The following two statements will imply our theorem. The first one provides us with cycles of every even length from a long interval. Our hope is that if the constants in the theorem are chosen appropriately then $2k$ falls into this interval.

For the statement let us recall that the *radius* $rad(H)$ of a graph H is the distance from a "central vertex" of H to a farthest vertex. Formally,

$$rad(H) = \min_{v \in V(H)} \max_{x \in V(H)} dist_H(v,x),$$

where $dist_H(v,x)$ is the length of the shortest v,x-path in H.