Constructive Combinatorics — Summer 2018
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Exercise Sheet 1

Due date: Apr 25th, 12:30 PM

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1. Show that for any k, the n-vertex K_k-free graph with the maximum number of copies of K_3 is the Turán graph $T_{n,k-1}$. (*Hint:* You can try apply the Zykov symmetrization: for non-adjacent vertices $u, v \in V(G)$ with $d(u) \geq d(v)$, let G' be the graph obtained from G by deleting all edges between v and $N(v)$ and adding all edges between v and $N(u)$.)

Remark. One can generalise this by replacing 'K_3' with 'K_t' for any t.

Exercise 2. For the octahedron, $K_{2,2,2}$, show that for all $n \geq 3$ we have the *strict* inequality $\mathrm{ex}(n, K_{2,2,2}) > e(T_{n,2})$. How large an n-vertex octahedron-free graph can you find?

Exercise 3. Show that for any tree T with t edges, $\frac{(t-1)n}{2} - o(n) \leq \mathrm{ex}(n, T) \leq (t-1)n$. In the special case of the star graph, $T = K_{1,t}$, show that the lower bound is correct.

Exercise 4. Give a Regularity Lemma-free proof of the Erdős-Stone Theorem.

(a) Show that for every $r, s \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, $0 < \epsilon < 1/r$ there exists $t \in \mathbb{N}$, $\delta > 0$ and N_0 such that for any graph G on $n > N_0$ vertices and with minimum degree $\delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n$, and any r pairwise disjoint t-element subsets $B_1, \ldots, B_r \subseteq V(G)$, the set

$$W = \{w \in V(G) \setminus (B_1, \cup \cdots \cup B_r) : |N(w, B_i)| \geq s \text{ for all } i = 1, \ldots r\}$$

has size at least δn.

(b) Use (a) to show that for any $\epsilon > 0$ and integers $r \geq 2$, $s \geq 1$ there exists an integer $N = N(r, s, \epsilon)$, such that any graph G on $n \geq N$ vertices and with $\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon\right) n$ contains $T_{rs,r}$.

(c) For every $r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, $0 < \epsilon < 1/r$, there exists a $\delta = \delta(r, \epsilon)$ and $M = M(r, \delta)$ such that, for all graphs G with $n \geq M$ vertices and at least $\left(1 - \frac{1}{r-1} + \epsilon\right) \frac{n^2}{2}$ edges there is a subgraph $H \subseteq G$ with at least δn vertices, where each vertex has at least $\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right) v(H)$ neighbors.

(d) Conclude the Erdős-Stone Theorem.